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UE NANOPHOTONICS

Quantum states of light:
Photon-photon correlations and Characteristic functions

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Statistical variations of photon number

Fluctuations of measurement of an observable, $\hat{O}$: $\langle \hat{O} \rangle = \mathrm{tr}\{\hat{\rho}\hat{O}\}$

$$(\Delta O)^2 \equiv \left\langle \left(\hat{O} - \langle \hat{O} \rangle\right)^2 \right\rangle = \langle \hat{O}^2 \rangle - \left\langle 2\hat{O}\langle \hat{O} \rangle \right\rangle + \langle \hat{O} \rangle^2$$

$$= \langle \hat{O}^2 \rangle - 2\langle \hat{O} \rangle^2 + \langle \hat{O} \rangle^2$$

$$= \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$$

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$$

Fluctuations of the number operator :

Since $I \propto \bar{n}$

$$(\Delta n)^2 = \langle \widehat{N}^2 \rangle - \langle \widehat{N} \rangle^2 \qquad \widehat{N} = \hat{a}^\dagger \hat{a} \qquad \langle \widehat{N} \rangle \longrightarrow \hat{I}$$

$$\langle \widehat{N}^2 \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle$$

$$[\hat{a}, \hat{a}^\dagger] = 1 \longrightarrow \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1 \longrightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1$$

$$\langle \widehat{N}^2 \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) \hat{a} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle \qquad \langle \widehat{:I:^2} \rangle \equiv \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle$$

$$= \langle \hat{a}^\dagger \hat{a} \rangle + \langle : \hat{a}^\dagger \hat{a} :^2 \rangle = \langle \hat{I} \rangle + \langle \widehat{:I:^2} \rangle$$

In the quantum description of light $\langle \widehat{:I:^{\mathbf{2}}} \rangle \propto \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle \equiv \langle : \hat{a}^\dagger \hat{a} :^2 \rangle$

$$(\Delta n)^2 = \langle \widehat{N}^2 \rangle - \langle \widehat{N} \rangle^2 \longrightarrow \langle : \hat{a}^\dagger \hat{a} :^2 \rangle + \langle \hat{I} \rangle - \langle \hat{I} \rangle^2 = \langle \widehat{:I:^2} \rangle + \langle \hat{I} \rangle - \langle \hat{I} \rangle^2 = \langle \hat{I} \rangle + (\Delta I)^2$$

Fluctuations of photon number :

$$\langle : \hat{a}^\dagger \hat{a} :^2 \rangle = \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle = \langle \hat{I}^2 \rangle$$

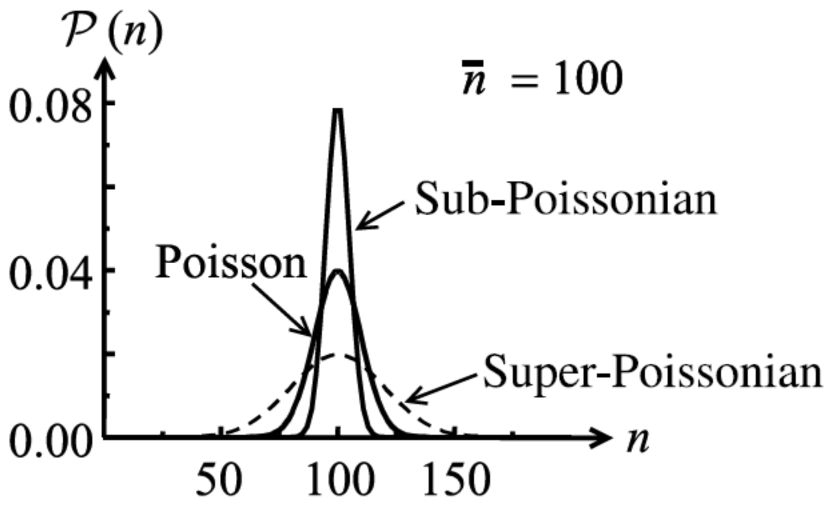
$$(\Delta n)^2 = \langle : \hat{a}^\dagger \hat{a} :^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{N} \rangle^2 = \langle \hat{N} \rangle + \langle : \hat{a}^\dagger \hat{a} :^2 \rangle - \langle \hat{n} \rangle^2$$

$$= \langle \hat{N} \rangle + \langle : \hat{I} :^2 \rangle - \langle \hat{I} \rangle^2$$

$$= \langle \hat{N} \rangle + (\Delta I)^2$$

$$(\Delta I)^2 = \langle : \hat{I} :^2 \rangle - \langle \hat{I} \rangle^2$$

$$(\Delta n)^2 = \langle \hat{I} \rangle + \begin{cases} (\Delta I)^2 > 0 & \text{Super-poissonian} \\ (\Delta I)^2 = 0 & \text{Poissonian} \\ (\Delta I)^2 < 0 & \text{Sub-poissonian} \end{cases}$$



Fluctuations of photon number : $(\Delta I)^2 = \langle I^2 \rangle - I^2$

$$(\Delta n)^2 = \langle \hat{N} \rangle + \begin{cases} (\Delta I)^2 > 0 & \text{Super-Poissonian} \\ (\Delta I)^2 = 0 & \text{Poissonian} \\ (\Delta I)^2 < 0 & \text{Sub-Poissonian} \end{cases}$$

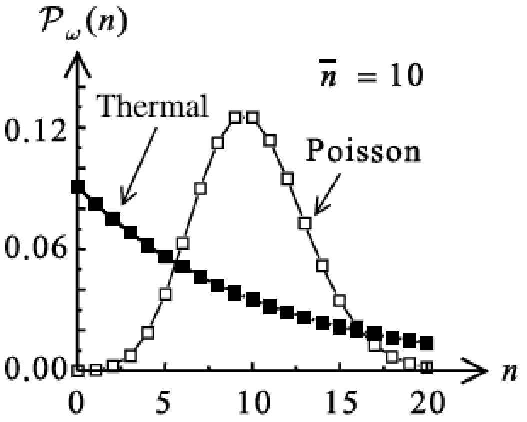
- Super-Poissonian → Classical `chaotic light', black-body radiation certain kinds of quantum light
- Poissonian → Coherent `Laser' light
- Sub-Poissonian → Requires `quantum' light

Thermal 'Black-body' radiation: $(\Delta I)^2 = \bar{n}^2$

Poissonian 'laser' light: $(\Delta I)^2 = 0$

Number state light: $(\Delta I)^2 = -n$

(*i.e.* $\Delta n = 0$)

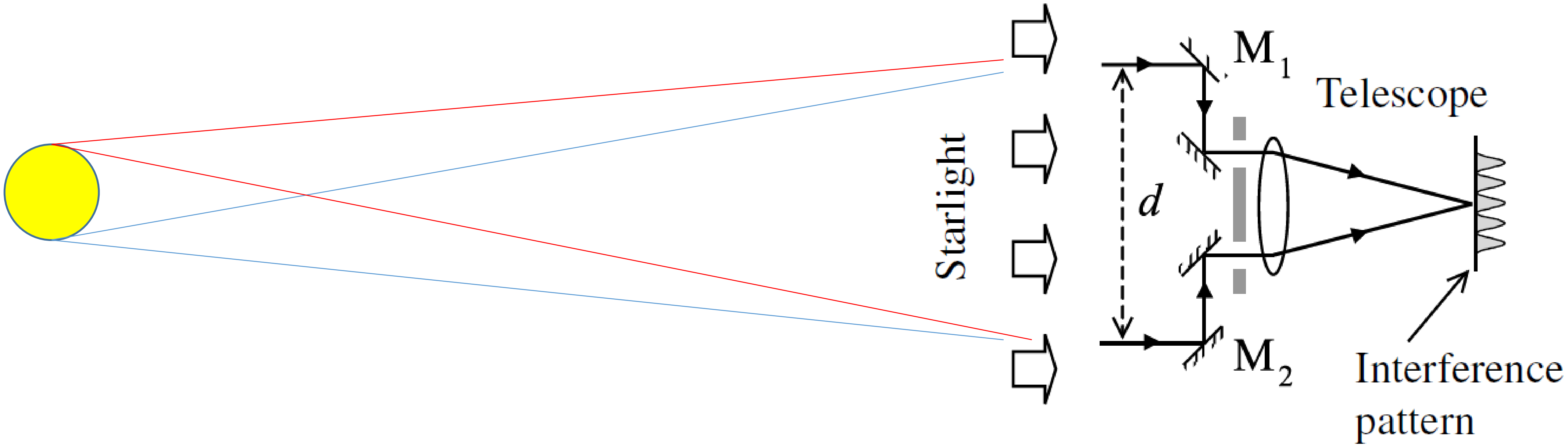


Black-body: $\mathcal{P}_{\text{Th}}(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$

Poisson : $\mathcal{P}_{\text{Poiss}}(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$

Photon Correlations

Michaelson stellar interferometry :



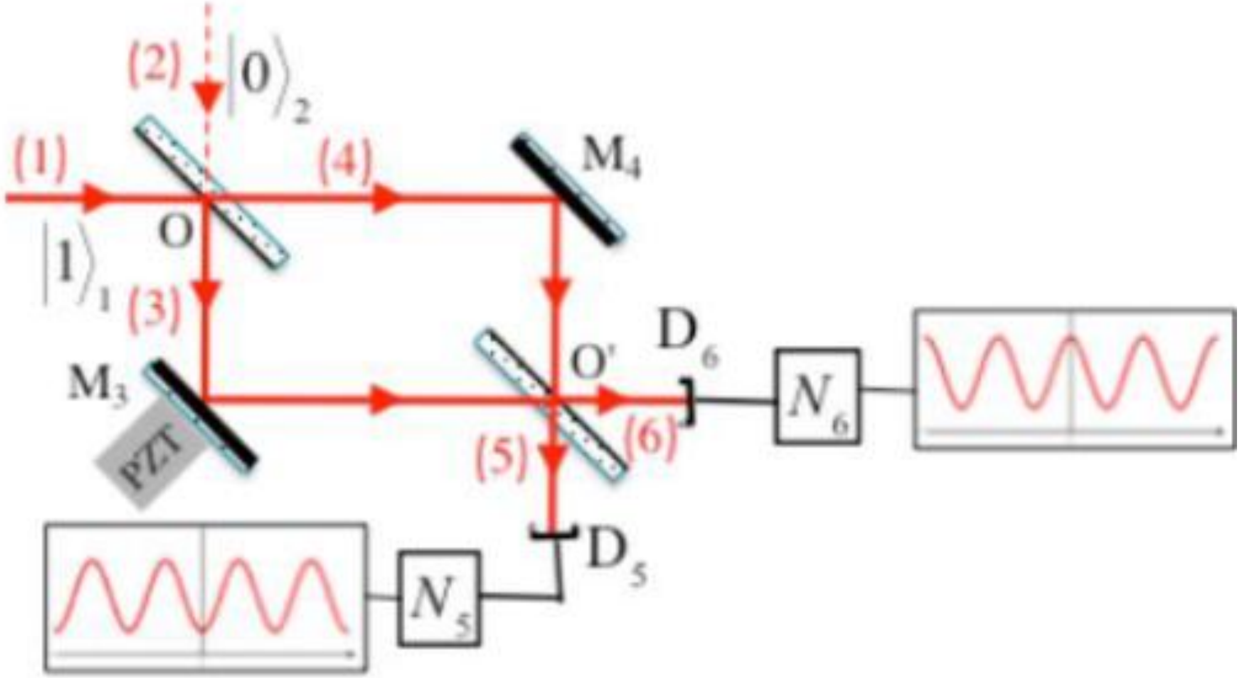
Interference fringes allow to determine angula size of the star

Drawbacks :

- Length of *d* is limited
- Sensitive to atmospheric disturbances

Mach-Zhender interferometer simulates the stellar interferometer physics

$$\langle f \rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$$



We ignore the space time−dependence, χ, until we need it

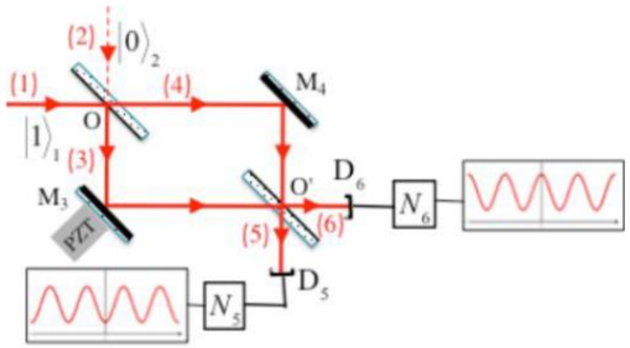
We ignore the operator nature of the field until we need it

Let us designate the fields at one of the detectors as :

$$E(\chi) = E_u(\chi_u) + E_d(\chi_u) = \left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right)$$

$$\langle I(\chi_u, \chi_d)\rangle = \langle E^2 \rangle = \left\langle \left\{\left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right)\right\}\left\{\left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right)\right\}\right\rangle$$

Mach-Zhender interferometer measures field correlations



$$E(\chi) = \left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right)$$

$$\langle I(\chi_u, \chi_d)\rangle = \langle E^2 \rangle = \left\langle \left\{ \left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right) \right\} \left\{ \left(E_u^{(+)} + E_u^{(-)}\right) + \left(E_d^{(+)} + E_d^{(-)}\right) \right\} \right\rangle$$

$$= \left\langle E_u^{(+)} E_u^{(+)} \right\rangle + \left\langle E_u^{(+)} E_u^{(-)} \right\rangle + \left\langle E_u^{(+)} E_d^{(+)} \right\rangle + \left\langle E_u^{(+)} E_d^{(-)} \right\rangle$$

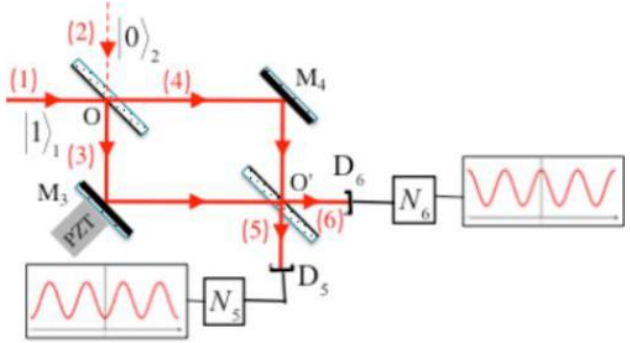
$$+ \left\langle E_u^{(-)} E_u^{(+)} \right\rangle + \left\langle E_u^{(-)} E_u^{(-)} \right\rangle + \left\langle E_u^{(-)} E_d^{(+)} \right\rangle + \left\langle E_u^{(-)} E_d^{(-)} \right\rangle$$

$$+ \left\langle E_d^{(+)} E_u^{(+)} \right\rangle + \left\langle E_d^{(+)} E_u^{(-)} \right\rangle + \left\langle E_d^{(+)} E_d^{(+)} \right\rangle + \left\langle E_d^{(+)} E_d^{(-)} \right\rangle$$

$$+ \left\langle E_d^{(-)} E_u^{(+)} \right\rangle + \left\langle E_d^{(-)} E_u^{(-)} \right\rangle + \left\langle E_d^{(-)} E_d^{(+)} \right\rangle + \left\langle E_d^{(-)} E_d^{(-)} \right\rangle$$

All terms in **red** average to zero over the detection time

Mach-Zhender signal averages over the field correlations : $g^{(1)}$



$$\langle I(\chi_u,\chi_d)\rangle = \left\langle E_u^{(+)}E_u^{(-)}\right\rangle + \left\langle E_u^{(-)}E_u^{(+)}\right\rangle + \left\langle E_u^{(+)}E_d^{(-)}\right\rangle + \left\langle E_u^{(-)}E_d^{(+)}\right\rangle$$

$$+ \left\langle E_d^{(+)}E_u^{(-)}\right\rangle + \left\langle E_d^{(-)}E_u^{(+)}\right\rangle + \left\langle E_d^{(+)}E_d^{(-)}\right\rangle + \left\langle E_d^{(-)}E_d^{(+)}\right\rangle$$

$$= 2\left(\langle I_u\rangle + \langle I_d\rangle + \left\langle E_u^{(+)}E_d^{(-)}\right\rangle + \left\langle E_u^{(-)}E_d^{(+)}\right\rangle\right)$$

$$= 2\left(\langle I_u\rangle + \langle I_d\rangle + 2\mathrm{Re}\left\{\left\langle E_u^{(-)}E_d^{(+)}\right\rangle\right\}\right)$$

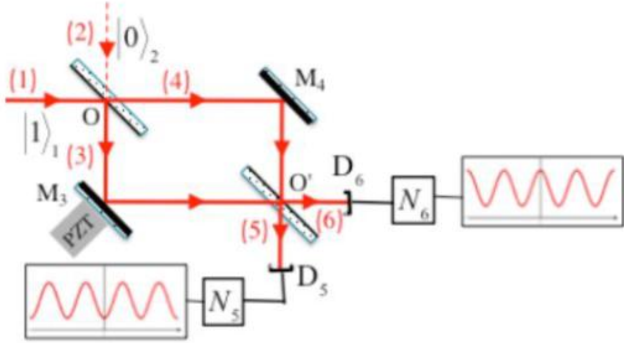
$$\left\langle E_u^{(-)}E_d^{(+)}\right\rangle \to G^{(1)}(\chi_u,\chi_d)$$

$$= 2(\langle I_u\rangle + \langle I_d\rangle)\left(1 + \frac{2\sqrt{\langle I_u\rangle\langle I_d\rangle}}{\langle I_u\rangle + \langle I_d\rangle}\mathrm{Re}\left\{\frac{\left\langle E_u^{(-)}E_d^{(+)}\right\rangle}{\sqrt{\langle I_u\rangle\langle I_d\rangle}}\right\}\right)$$

$$g^{(1)}(\chi_u,\chi_d) \equiv \frac{\left\langle E_u^{(-)}E_d^{(+)}\right\rangle}{\sqrt{\langle I_u\rangle\langle I_d\rangle}}$$

Normalized first order correlation function, $g^{(1)}$, gives us the fringe Visibility, V

$$\left|g^{(1)}(\chi_u, \chi_d)\right| \Longrightarrow V \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$



For a Max-Zhender interferometer :

$$g^{(1)}(\chi_u, \chi_d) \equiv \frac{\left\langle E_u^{(-)} E_d^{(+)} \right\rangle}{\sqrt{\langle I_u \rangle \langle I_d \rangle}}$$

$$E_u^{(-)} \longrightarrow e^{\left(i\boldsymbol{k}\cdot\boldsymbol{r}_u + i\phi_u(\boldsymbol{r},t)\right)} \qquad E_d^{(+)} \longrightarrow e^{\left(-i\boldsymbol{k}\cdot\boldsymbol{r}_d - i\phi_d(\boldsymbol{r},t)\right)}$$

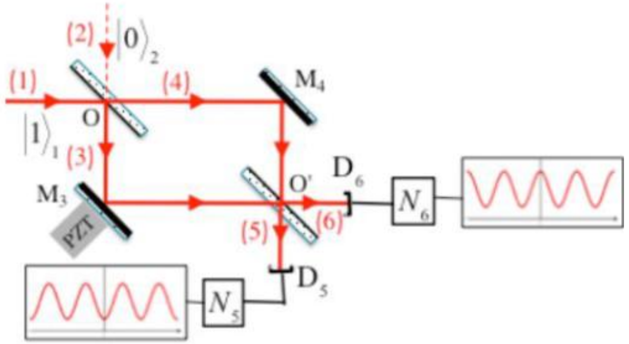
$$g^{(1)}(\chi_u, \chi_d) \sim \left\langle \exp\left(i\boldsymbol{k}\cdot(\boldsymbol{r}_u - \boldsymbol{r}_d) + i\left(\phi_u(\boldsymbol{r}_u, t) - \phi_d(\boldsymbol{r}_d, t)\right)\right)\right\rangle$$

Visibility of interference fringes is affected by the stability of $\left(\phi_u(\boldsymbol{r}_u, t) - \phi_d(\boldsymbol{r}_d, t)\right)$ over time

Since the light comes from the same source, we call $g^{(1)}$ an auto-correlation function

Spatial dependence of $g^{(1)}$ corresponds to autocorrelations in time

$$g^{(1)}(\chi_u,\chi_d) \equiv \frac{\left\langle E_u^{(-)} E_d^{(+)} \right\rangle}{\sqrt{\langle I_u \rangle \langle I_d \rangle}} \qquad g^{(1)}(\chi_u,\chi_d) \leq 1$$



$$g^{(1)}(\chi_u,\chi_d) \equiv \frac{\left\langle E_u^{(-)} E_d^{(+)} \right\rangle}{\sqrt{\langle I_u \rangle \langle I_d \rangle}} \rightarrow \langle \exp(ik(l_u - l_d) + i(\phi_u - \phi_d)) \rangle$$

$$\text{or} \quad \left\langle \exp\left(i\frac{\omega}{c}(l_u - l_d) + i(\phi_u - \phi_d) \right) \right\rangle$$

$$E_u^{(-)} \longrightarrow e^{(ikl_u + i\phi_u(\boldsymbol{r},t))}$$

$$E_d^{(+)} \longrightarrow e^{(-ikl_d - i\phi_d(\boldsymbol{r},t))}$$

Difference in path length corresponds to differences in time so , $g^{(1)}\left(\tau = \frac{(l_u - l_d)}{c}\right)$, is also termporal auto−correlation

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\sqrt{\langle E^*(t)E(t)\rangle}} \propto G^{(1)}(\tau) = \langle E^*(t)E(t+\tau)\rangle$$

Physical significance of $g^{(1)}(\tau)$: Fourier transform

$$G^{(1)}(\tau) = \langle E^*(t)E(t+\tau)\rangle = \int_{-\infty}^{\infty} dt E^*(t)E(t+\tau)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \tilde{E}^*(\omega)\, \tilde{E}(\omega') e^{-i\omega\tau} e^{i\omega t} e^{-i\omega' t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau} \int_{-\infty}^{\infty} d\omega'\, \tilde{E}^*(\omega)\tilde{E}(\omega') \int_{-\infty}^{\infty} dt e^{i(\omega-\omega')t}$$

$$= \int_{-\infty}^{\infty} d\omega\, \tilde{E}^*(\omega)\tilde{E}(\omega) e^{-i\omega\tau}$$

$$= \int_{-\infty}^{\infty} d\omega\, S(\omega) e^{-i\omega\tau}$$

$$E(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \tilde{E}(\omega) e^{-i\omega t}$$

$$\int_{-\infty}^{\infty} dt e^{i(\omega-\omega')t} = 2\pi\delta(\omega-\omega')$$

$$S(\omega) = \left|\tilde{E}(\omega)\right|^2$$

$S(\omega)$ is the power spectrum of the beam

For a normalized field : $g^{(1)}(\tau) = \int_{-\infty}^{\infty} d\omega\, S(\omega) e^{-i\omega\tau}$

The power spectrum can be measured with spectrometers and then you deduce $g^{(1)}(\tau)$ from the Fourier transform of $S(\omega)$

Perfectly coherent beam : $S(\omega) = \delta(\omega - \omega_0)$

Physical significance of $g^{(1)}(\tau)$: Wiener-Kintchine theorem

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, g^{(1)}(\tau)e^{i\omega\tau} = \frac{1}{2\pi}\left\{\int_{-\infty}^{0} d\tau + \int_{0}^{\infty} d\tau\right\} g^{(1)}(\tau)e^{i\omega\tau}$$

$\tau \to -\tau$

$$= \frac{1}{2\pi}\left\{\int_{\infty}^{0} g^{(1)}(-\tau)e^{-i\omega\tau} d(-\tau) + \int_{0}^{\infty} g^{(1)}(\tau)e^{i\omega\tau} d\tau\right\}$$

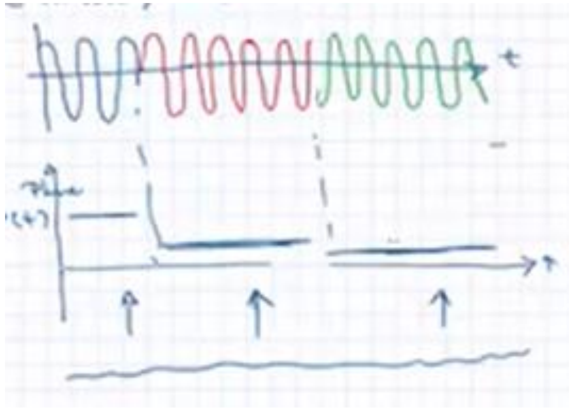
$$g^{(1)}(-\tau) = \left(g^{(1)}(\tau)\right)^*$$

$$= \frac{1}{2\pi}\left\{\int_{\infty}^{0} \left(g^{(1)}(\tau)e^{i\omega\tau}\right)^* d(-\tau) + \int_{0}^{\infty} g^{(1)}(\tau)e^{i\omega\tau} d\omega\right\}$$

$$= \frac{1}{\pi}\mathrm{Re}\left\{\int_{0}^{\infty} d\tau\, g^{(1)}(\tau)e^{i\omega\tau}\right\}$$

Wiener Kintchine theorem

Collisional “broadening” as described by $g^{(1)}(\tau)$



$$E(t) = E_0 \sum_{j=1}^{N} e^{-i\omega_0 t + i\phi_j(t)} \qquad G^{(1)}(\tau) = \langle E^*(t) E(t+\tau) \rangle$$

$$G^{(1)}(\tau) = \left\langle |E_0|^2 \sum_{j=1}^{N} \sum_{k=1}^{N} e^{i\omega_0 t - i\phi_j(t)} e^{-i\omega_0 (t+\tau) + i\phi_k(t+\tau)} \right\rangle = |E_0|^2 e^{-i\omega_0 \tau} \left\langle \sum_{j=1}^{N} \sum_{k=1}^{N} e^{i\left(\phi_k(t+\tau) - \phi_j(t)\right)} \right\rangle$$

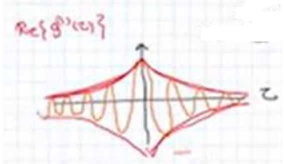
When $k \neq j$ $\quad \left\langle e^{i\left(\phi_k(t+\tau) - \phi_j(t)\right)} \right\rangle = 0$

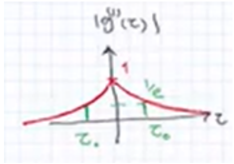
$$P(\tau) d\tau = \frac{1}{\tau_0} e^{-\frac{\tau}{\tau_0}} d\tau$$

$$G^{(1)}(\tau) = |E_0|^2 e^{-i\omega_0 \tau} N \left\langle \sum_{j=1}^{N} e^{i\left(\phi_j(t+\tau) - \phi_j(t)\right)} \right\rangle$$

$$\left\langle \sum_{j=1}^{N} e^{i\left(\phi_j(t+\tau) - \phi_j(t)\right)} \right\rangle = \int_{\tau}^{\infty} P(\tau) d\tau = \frac{1}{\tau_0} \int_{\tau}^{\infty} e^{-\frac{\tau}{\tau_0}} d\tau = e^{-\frac{\tau}{\tau_0}}$$

$$g^{(1)}(\tau) = e^{-i\omega_0 \tau} e^{-\frac{|\tau|}{\tau_0}}$$





$$S(\omega) = \frac{1}{\pi} \mathrm{Re} \left\{ \int_0^{\infty} d\tau \, g^{(1)}(\tau) e^{i\omega\tau} \right\} = \frac{1}{\tau_0 \pi} \frac{1}{\left(\frac{1}{\tau_0}\right)^2 + (\omega - \omega_0)^2}$$

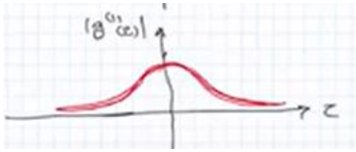
Doppler "broadening" described by $g^{(1)}(\tau)$

Phases are taken as time independent, but frequencies have thermally distributed Doppler shifts

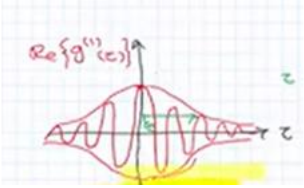
$$E(t) = E_0 \sum_{j=1}^{N} e^{-i\omega_j t + i\phi_j} \qquad G^{(1)}(\tau) = \langle E^*(t)E(t+\tau)\rangle$$

$$G^{(1)}(\tau) = \left\langle |E_0|^2 \sum_{j=1}^{N}\sum_{k=1}^{N} e^{i\omega_j t - i\phi_j} e^{-i\omega_k(t+\tau)+i\phi_k} \right\rangle = |E_0|^2 e^{-i\omega_0\tau} \left\langle \sum_{j=1}^{N}\sum_{k=1}^{N} e^{i(\omega_j-\omega_k)t + i(\phi_k-\phi_j) - i\omega_k\tau} \right\rangle$$

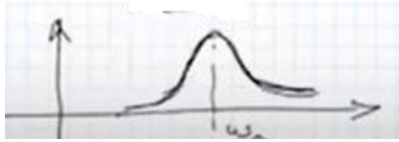
$$= |E_0|^2 \left\langle \sum_{j=1}^{N}\sum_{k=1}^{N} e^{i(\omega_j-\omega_k)t - i\omega_k\tau} \right\rangle = |E_0|^2 \left\langle \sum_{j=1}^{N} e^{-i\omega_j\tau} \right\rangle$$



$$= |E_0|^2 N \left\langle \sum_{j=1}^{N} e^{-i\omega_j\tau} \right\rangle = |E_0|^2 N \frac{1}{\Delta\sqrt{2\pi}} \int d\omega\, e^{-i\omega\tau} e^{-\frac{(\omega-\omega_0)^2}{2\Delta^2}} = |E_0|^2 N e^{-i\omega_0\tau} e^{-\frac{1}{2}\tau^2\Delta^2}$$



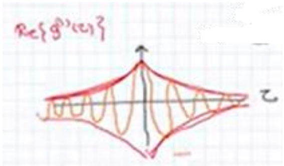
$$S(\omega) = \frac{1}{\pi}\mathrm{Re}\left\{\int_0^\infty d\tau\, g^{(1)}(\tau) e^{i\omega\tau}\right\} = \frac{1}{\Delta\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\omega-\omega_0)^2}{2\Delta^2}}$$



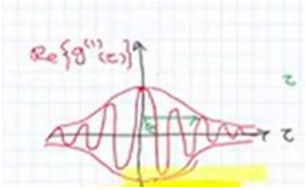
$$\Delta\tau \sim \frac{\sqrt{2}}{\Delta}$$

The classical light distributions : $g^{(1)}(\tau)$

- « Thermal light » (collisional broadening) $g^{(1)}(\tau) = e^{-i\omega_0\tau} e^{-\frac{\tau}{\tau_0}}$



- Doppler broadening (incoherent broadening) $g^{(1)}(\tau) = e^{-i\omega_0\tau} e^{-\frac{1}{2}\tau^2\Delta^2}$



- Coherent light (laser) $E(t) \sim E_0 e^{-i\omega_0 t + i\phi}$ $g^{(1)}(\tau) = e^{-i\omega_0\tau}$

Classical

Quantum

$$g^{(1)}(0) = \frac{\langle E^*(t)E(t)\rangle}{\langle E^*(t)E(t)\rangle} = \frac{\langle : \hat{a}^\dagger(t)\hat{a}(t) : \rangle}{\langle \hat{a}^\dagger \hat{a}\rangle} = 1$$

Classical | Quantum

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle^{1/2}\langle E^*(t+\tau)E(t+\tau)\rangle^{1/2}} = \frac{\langle : \hat{a}^\dagger(t)\hat{a}(t+\tau) : \rangle}{\langle \hat{a}^\dagger \hat{a}\rangle}$$

$: \hat{a}^\dagger \hat{a} :$ Is called a "normal ordered" operator

Michelson stellar interferometer: $g^{(1)}(\tau)$

We can do a classical calculation since it gives the same results as the quantum calculation

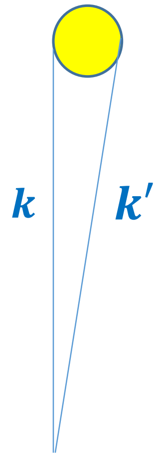
$$E_t \longrightarrow \mathcal{E}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1} + \mathcal{E}_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_1} + \mathcal{E}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_2} + \mathcal{E}_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_2}$$

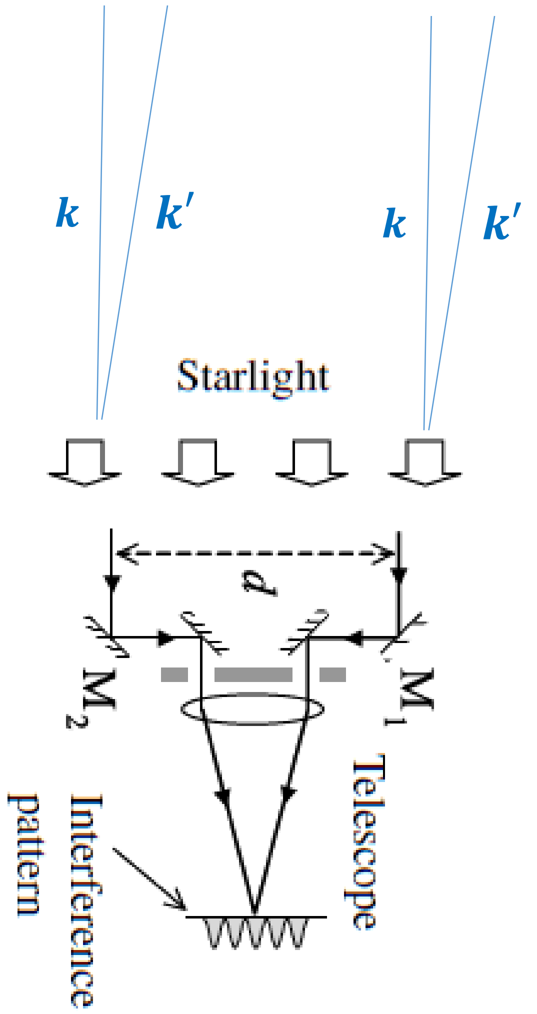
$$\langle \mathcal{E}_{\boldsymbol{k}} \mathcal{E}_{\boldsymbol{k}'} \rangle = 0$$

$$\langle I \rangle = \eta \langle E_t^* E_t \rangle = \eta \langle 2(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2) \rangle$$

$$+\eta \langle 2\left(|\mathcal{E}_{\boldsymbol{k}}|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} + \mathrm{c.c.}\right) \rangle$$

$$+\eta \left\langle 2\left(|\mathcal{E}_{\boldsymbol{k}'}|^2 e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} + \mathrm{c.c.}\right) \right\rangle$$





Michelson stellar interferometer: $g^{(1)}(\tau)$

$$\langle I\rangle = \eta\langle E_t^* E_t\rangle = \eta\langle 2(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2)\rangle + \eta\left\langle 2\left(|\mathcal{E}_{\boldsymbol{k}}|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)} + \text{c.c.}\right)\right\rangle + \eta\left\langle 2\left(|\mathcal{E}_{\boldsymbol{k}'}|^2 e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)} + \text{c.c.}\right)\right\rangle$$

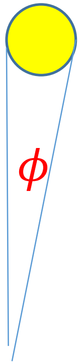
Assuming equal intensities from the two sides of the star : $|\mathcal{E}_{\boldsymbol{k}}|^2 = |\mathcal{E}_{\boldsymbol{k}'}|^2 = I_0$

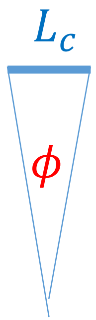
$$\langle I\rangle = 4\eta\left(1 + \cos\left(\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + \cos\left(\boldsymbol{k}'\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\right)$$

$$\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b) \Rightarrow \cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\langle I\rangle = 4\eta\left(1 + \cos\left(\frac{\boldsymbol{k}+\boldsymbol{k}'}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + \cos\left(\frac{\boldsymbol{k}-\boldsymbol{k}'}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\right) = 4\eta\left(1 + \cos\left(\frac{\boldsymbol{k}+\boldsymbol{k}'}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\right)$$

$$\underbrace{\cos\left(\frac{\Delta\boldsymbol{k}}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)}_{\pi\frac{\phi}{\lambda}d}$$

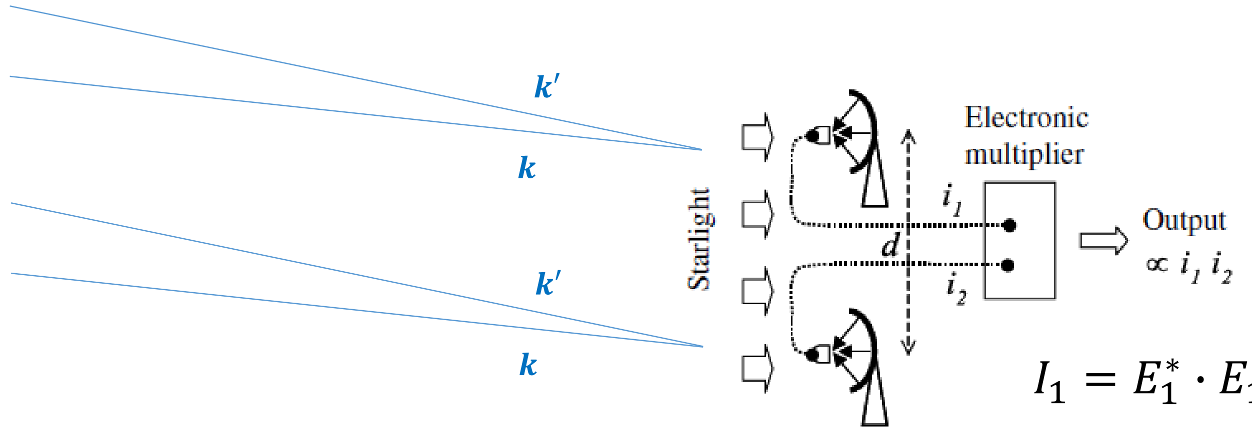




Fringe spacing : $\pi\frac{\phi}{\lambda}L_c = \pi \Rightarrow \phi = \frac{\lambda}{L_c}$

Michelson stellar interferometry is hampered by the rapidly varying $\cos\left(\frac{\boldsymbol{k}+\boldsymbol{k}'}{2}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)$ term !

The Hanburry Brown and Twiss stellar interferometer: $g^{(2)}(\tau)$



$$I_1 = E_1^* \cdot E_1 \longrightarrow \left(\mathcal{E}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1} + \mathcal{E}_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_1}\right)^* \left(\mathcal{E}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1} + \mathcal{E}_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_1}\right)$$

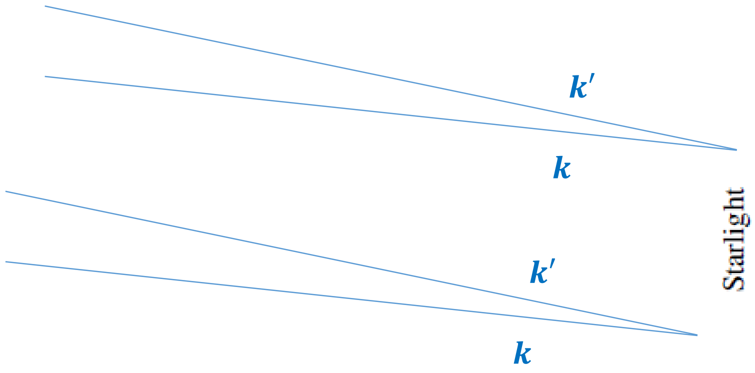
$$I(\boldsymbol{r}_1,t) = \eta\langle E_{1,t}^* E_{1,t}\rangle = \eta\left\langle\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2 + \mathcal{E}_{\boldsymbol{k}}^*\mathcal{E}_{\boldsymbol{k}'} e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}_1}\right) + \text{c.c.}\right\rangle \qquad I(\boldsymbol{r}_2,t) = \eta\langle E_{2,t}^* E_{2,t}\rangle = \eta\left\langle\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2 + \mathcal{E}_{\boldsymbol{k}}^*\mathcal{E}_{\boldsymbol{k}'} e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}_2}\right) + \text{c.c.}\right\rangle$$

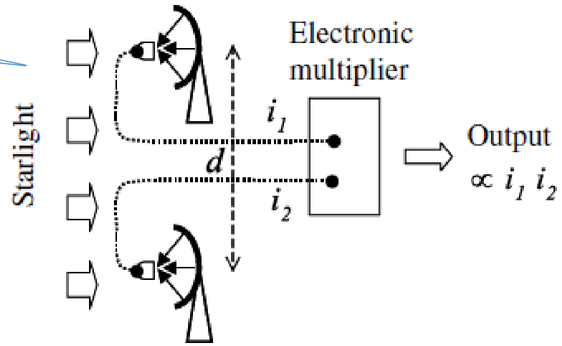
$$\langle I(\boldsymbol{r}_1,t) I(\boldsymbol{r}_2,t)\rangle = \eta^2\left\langle\left(\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)^2 + 2\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)\mathcal{E}_{\boldsymbol{k}}^*\mathcal{E}_{\boldsymbol{k}'}(\cdots)\right) + \mathcal{E}_{\boldsymbol{k}}^*\mathcal{E}_{\boldsymbol{k}}^*\mathcal{E}_{\boldsymbol{k}'}\mathcal{E}_{\boldsymbol{k}'} e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}_2} e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}_2} + \text{c.c.}\right\rangle$$

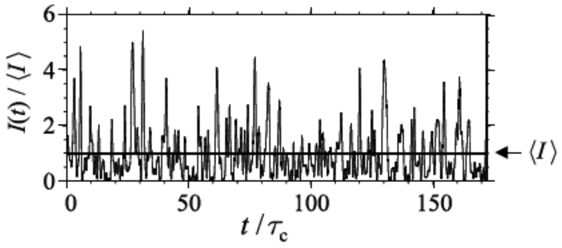
$$= \eta^2\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)^2\left(1 + \frac{2\langle\mathcal{E}_{\boldsymbol{k}}^2\rangle\langle\mathcal{E}_{\boldsymbol{k}'}^2\rangle}{\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)^2}\cos\left[\left(\frac{\boldsymbol{k}'-\boldsymbol{k}}{2}\right)(\boldsymbol{r}_1 - \boldsymbol{r}_2)\right]\right)$$

$$\langle I(\boldsymbol{r}_1,t) I(\boldsymbol{r}_2,t)\rangle \propto \left(1 + \frac{2\langle\mathcal{E}_{\boldsymbol{k}}^2\rangle\langle\mathcal{E}_{\boldsymbol{k}'}^2\rangle}{\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)^2}\cos\left[\left(\frac{\Delta\boldsymbol{k}}{2}\right)\cdot\Delta\boldsymbol{r}\right]\right) = \left(1 + \frac{2\langle\mathcal{E}_{\boldsymbol{k}}^2\rangle\langle\mathcal{E}_{\boldsymbol{k}'}^2\rangle}{\left(|\mathcal{E}_{\boldsymbol{k}}|^2 + |\mathcal{E}_{\boldsymbol{k}'}|^2\right)^2}\cos\left[\left(\frac{\pi L_c}{\lambda}\right)\phi\right]\right)$$

The Hanburry Brown and Twiss stellar interferometer: $g^{(2)}(\tau)$







The previous calculations were a special case for correlation functions of Gaussian random processes

$$g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau) \equiv \frac{\langle E^*(\boldsymbol{r}_1,t)E(\boldsymbol{r}_2,t+\tau)\rangle}{\langle |E(\boldsymbol{r}_1,t)|^2\rangle^{1/2}\langle |E(\boldsymbol{r}_2,t+\tau)|^2\rangle^{1/2}}$$

$$g^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau) \equiv \frac{\langle I(\boldsymbol{r}_1,t)I(\boldsymbol{r}_2,t+\tau)\rangle}{\langle I(\boldsymbol{r}_1,t)\rangle\langle I(\boldsymbol{r}_2,t+\tau)\rangle} = \frac{\langle E^*(\boldsymbol{r}_1,t)E(\boldsymbol{r}_1,t)E^*(\boldsymbol{r}_1,t+\tau)E(\boldsymbol{r}_2,t+\tau)\rangle}{\langle |E(\boldsymbol{r}_1,t)|^2\rangle\langle |E(\boldsymbol{r}_2,t+\tau)|^2\rangle}$$

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r};\tau) \longrightarrow \frac{\left\langle E^{(-)}(\boldsymbol{r},t)E^{(-)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t)\right\rangle}{\left\langle E^{(-)}(\boldsymbol{r},t)E^{(+)}(\boldsymbol{r},t)\right\rangle\left\langle E^{(-)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t+\tau)\right\rangle}$$

$$g^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau) = 1 + \left|g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau)\right|^2$$

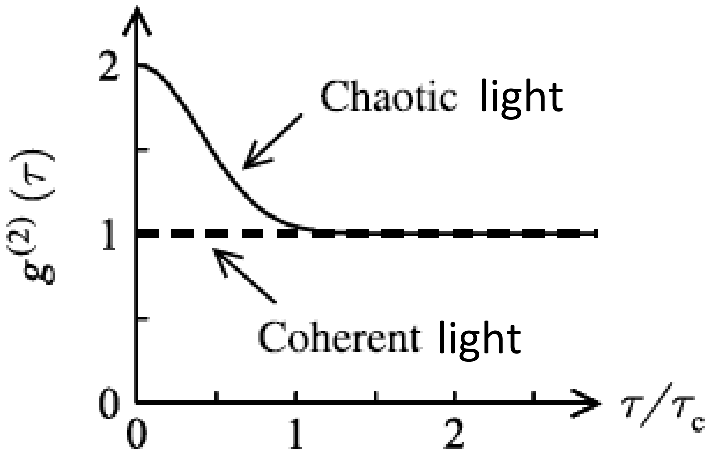
The Hanburry Brown and Twiss stellar interferometer: $g^{(2)}(\tau)$

$$g^{(2)}(\boldsymbol{r},\boldsymbol{r};\tau) \longrightarrow \frac{\left\langle E^{(-)}(\boldsymbol{r},t)E^{(-)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t)\right\rangle}{\left\langle E^{(-)}(\boldsymbol{r},t)E^{(+)}(\boldsymbol{r},t)\right\rangle\left\langle E^{(-)}(\boldsymbol{r},t+\tau)E^{(+)}(\boldsymbol{r},t+\tau)\right\rangle}$$

Consequence of classical fluctuations (central limit theorem):

$$g^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau) = 1 + \left|g^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_1;\tau)\right|^2$$

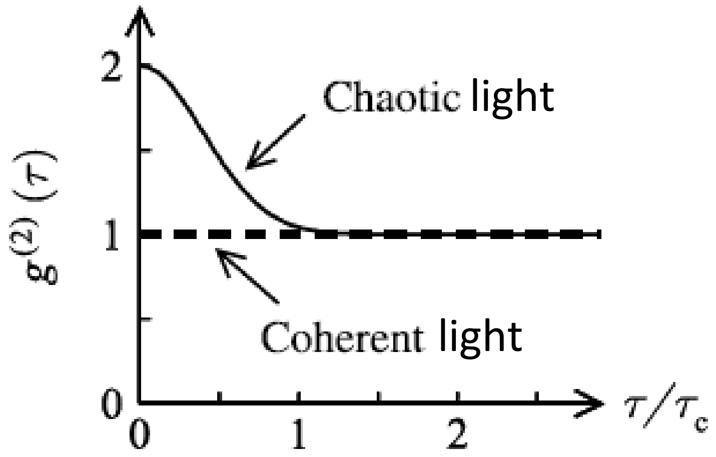
$$g^{(2)}(\boldsymbol{r},\boldsymbol{r};\tau) \geq 1$$



$$g^{(2)}(\boldsymbol{r},\boldsymbol{r};\infty) \to 1$$

Quantum correlation : $g_q^{(2)}(\tau)$

$$g_q^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2}$$

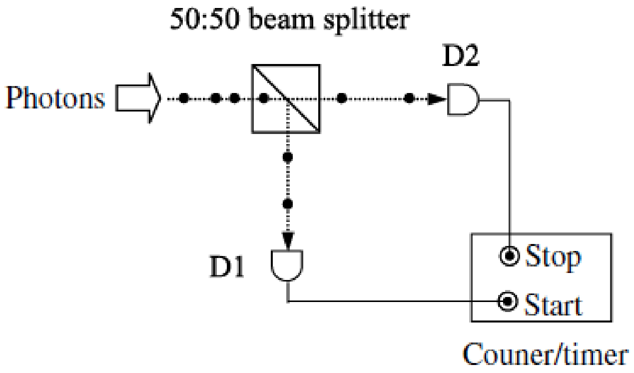


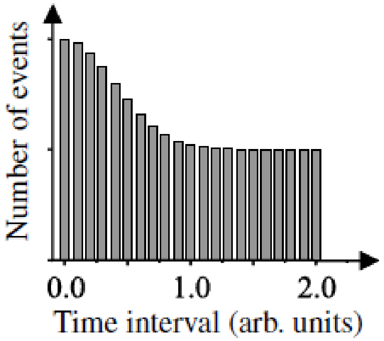
$$g^{(2)}(\infty) \to 1$$

Unlike $g^{(1)}(\tau)$ which is largely consistent with classical behavior,
the behavior of the normal ordered, $g_q^{(1)}(\tau)$ can be fundametally distinct from $g_{\mathrm{cl}}^{(1)}(\tau)$

Hanburry Brown & Twiss experiment : $g_{\mathrm{cl}}^{(1)}(\tau)$

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t)\hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2}$$





Hanburry Brown & Twiss experiment was done with a classical light source as a Proof of principal for measuring stellar diameters with intensity correlations.

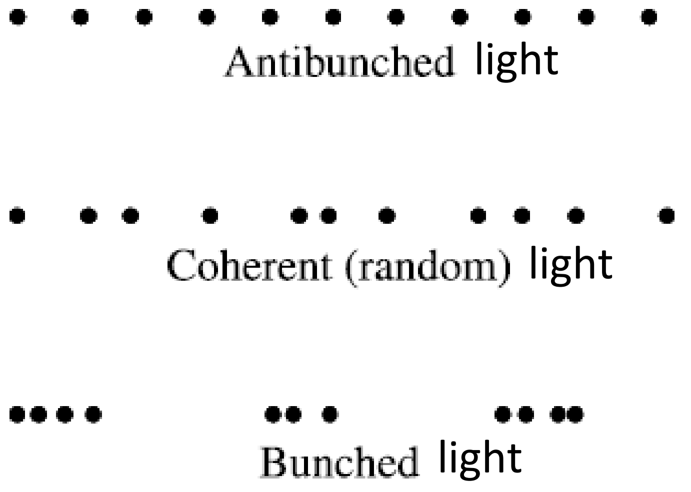
Unlike $g_{\mathrm{cl}}^{(2)}(0) \geq 1$, one can have $g_q^{(2)}(0) \leq 1$

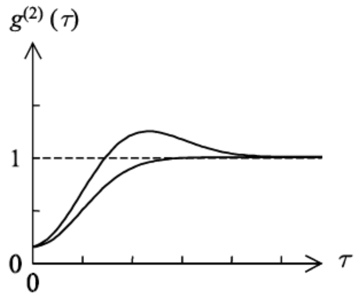
For a Fock state $|n\rangle$ $\quad g^{(2)}(0) = \dfrac{\langle n|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle}{\langle n|\hat{a}^\dagger\hat{a}|n\rangle^2} = \dfrac{\langle n|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|n\rangle - \langle n|\hat{a}^\dagger\hat{a}|n\rangle}{n^2}$

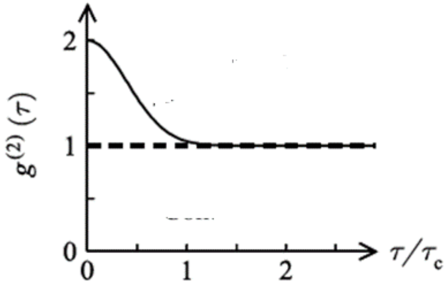
$$= \frac{n^2 - n}{n^2} = 1 - \frac{1}{n}$$

The highly "quantum" 1-photon state $|1\rangle$ has $g^{(2)}(0) = 0$

- $g^{(2)}(0)<1$ anti-bunched (quantum) light
- $g^{(2)}(0)=1$ coherent (laser) light
- $g^{(2)}(0)>1$ bunched (classical) light







Beware of misconceptions !

The properties $g^{(2)}(0) < 1$ and/or sub-Poissonian statistics both indicate quantum light !

But $g^{(2)}(0) > 1$ is not equivalent to super−Poissonian statistics and doesn′t necessarily imply "classical light"

The classical theory of $g^{(2)}$, $g^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_1; \tau) = 1 + \left|g^{(1)}(\boldsymbol{r}_1, \boldsymbol{r}_1; \tau)\right|^2$, predicts $g^{(2)}(0) = 2,$ for coherent light while quantum theory predicts the correct result of $g^{(2)}(0) = 1$ for coherent light.

Roy Glauber showed that the quantum theory also predicts $g^{(2)}(0) > 1$ for “chaotic light”

Pseudo-probability distributions

Density matrix : $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ $\quad \sum_i p_i = 1$

Expectation values : $\langle \hat{O} \rangle = \mathrm{Tr}\{\hat{\rho}\hat{O}\} = \mathrm{tr}\{\rho O\} = \sum_i p_i \langle\psi_i|\hat{O}|\psi_i\rangle$

Glauber-Sudarshan quasi-probability distribution, $P(\alpha)$:

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha \qquad \int P(\alpha) d^2\alpha = 1$$

$$\mathrm{Tr}\{\hat{\rho}\} = \int \sum_n P(\alpha)\langle n|\alpha\rangle\langle\alpha|n\rangle d^2\alpha = \int P(\alpha) \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle d^2\alpha = \int P(\alpha)\langle\alpha|\alpha\rangle d^2\alpha = \int P(\alpha) d^2\alpha = 1$$

Glauber-Sudarshan $P(\alpha)$ the determines the “quantumness” of a distribution :

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

$P(\alpha) > 0$ for all α $\Longrightarrow$ « classical » distributions of states

$P(\alpha) < 0$ for any α and/or more singular than a delta function $\Longrightarrow$ quantum light

Optical equivalence theorem :

Normal ordered operator : $:\hat{O}(\hat{a}^\dagger,\hat{a}):=\sum_n\sum_m C_{nm}(\hat{a}^\dagger)^n(\hat{a})^m$ $\hat{\rho}=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$

$$\langle:\hat{O}:\rangle=\mathrm{Tr}\{:O:\hat{\rho}\}=\mathrm{Tr}\left\{\int\sum_n\sum_m C_{nm}(\hat{a}^\dagger)^n(\hat{a})^m P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha\right\}$$

$$=\int P(\alpha)\sum_n\sum_m C_{nm}\langle\alpha|(\hat{a}^\dagger)^n(\hat{a})^m|\alpha\rangle d^2\alpha$$

$$=\int P(\alpha)\sum_n\sum_m C_{nm}\langle\alpha|(\alpha^*)^n(\alpha)^m|\alpha\rangle d^2\alpha$$

$$=\int P(\alpha)O^{(N)}(\alpha^*,\alpha)d^2\alpha$$

$$\langle:\hat{O}:\rangle=\int P(\alpha)O^{(N)}(\alpha^*,\alpha)d^2\alpha$$

Finding and using the $P(\alpha)$ function :

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle\, e^{\alpha u^* - u\alpha^*} d^2u$$

where $|u\rangle$ and $|-u\rangle$ are coherent states

$$\langle :\hat{O}: \rangle = \int P(\alpha) O^{(N)}(\alpha^*, \alpha) d^2\alpha$$

Example : Find $P_\beta(\alpha)$ for a coherent state $|\beta\rangle$: ⟹ $\hat{\rho} = |\beta\rangle\langle\beta|$

$$\langle -u|\beta\rangle = e^{-\frac{1}{2}|u|^2} e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-u^*)^m}{\sqrt{m!}}\frac{\beta^n}{\sqrt{n!}}\langle m|n\rangle = e^{-\frac{1}{2}|u|^2} e^{-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\frac{(-u^*\beta)^n}{n!} = e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - u^*\beta}$$

$$\langle\beta|u\rangle = e^{-\frac{1}{2}|u|^2 - \frac{1}{2}|\beta|^2 + u\beta^*}$$

$$P_\beta(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\beta\rangle\langle\beta|u\rangle e^{\alpha u^* - u\alpha^*} d^2u = \frac{e^{|\alpha|^2}e^{-|\beta|^2}}{\pi^2}\int e^{u^*(\alpha-\beta) - u(\alpha^* - \beta^*)} d^2u = \delta^2(\alpha - \beta)$$

Since $\delta^2(\alpha-\beta) = \delta\big(\mathrm{Re}(\alpha) - \mathrm{Re}(\beta)\big)\delta\big(\mathrm{Im}(\alpha) - \mathrm{Im}(\beta)\big) = \frac{1}{\pi^2}\int e^{u^*(\alpha-\beta) - u(\alpha^*-\beta^*)} d^2u$

Using the $P_\beta(\alpha)$ function for coherent light correlation :

$P_\beta(\alpha) = \delta^2(\alpha - \beta)$ for a coherent state $|\beta\rangle$

$$\bar{n} = \langle \hat{N} \rangle = \langle : \hat{a}^\dagger \hat{a} : \rangle = \int P_\beta(\alpha) N(\alpha^*, \alpha) d^2\alpha = \int \delta^2(\alpha - \beta)|\alpha|^2 d^2\alpha = |\beta|^2$$

$$g^{(2)}(0) = \frac{\langle\beta|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\beta\rangle}{\langle\beta|\hat{a}^\dagger\hat{a}|\beta\rangle^2} = \frac{\left\langle : \left(\hat{a}^\dagger\hat{a}\right)^2 : \right\rangle}{\langle : \hat{a}^\dagger\hat{a} : \rangle} = \frac{\int P(\alpha)|\alpha|^4 d^2\alpha}{\left(\int P(\alpha)|\alpha|^2 d^2\alpha\right)^2}$$

$$= \frac{\int \delta^2(\alpha - \beta)|\alpha|^4 d^2\alpha}{|\beta|^4} = 1$$

Much different than the naïve classical prediction of "$g^{(2)}(0) = 2$"

$P_{\mathrm{Th}}(\alpha)$ function for thermal light :

$$P_{\mathrm{Th}}(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}_{\mathrm{Th}}|u\rangle\, e^{\alpha u^* - u\alpha^*} d^2u$$

$$\hat{\rho}_{\mathrm{Th}} = \frac{e^{-\beta\omega\hat{a}^\dagger\hat{a}}}{Z}$$

$$\beta = \frac{\hbar}{k_B T}$$

$$Z = \frac{1}{1 - e^{-\beta\omega}}$$

Easier derivation and more instructive to put $\hat{\rho}_{\mathrm{Th}}$ in the form : $\hat{\rho}_{\mathrm{Th}} = \int P_{\mathrm{Th}}(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$

$$\hat{\mathbb{I}} = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|$$

$$\hat{\rho}_{\mathrm{Th}} = \frac{1}{Z}e^{-\beta\omega\hat{a}^\dagger\hat{a}} = \frac{1}{Z}e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}} = \frac{1}{\pi Z}\int d^2\alpha\, e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}|\alpha\rangle\langle\alpha|\, e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}$$

Put $\hat{\rho}_{\mathrm{Th}}$ in the form : $\hat{\rho}_{\mathrm{Th}} = \int P_{\mathrm{Th}}(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$

$$\hat{\rho}_{\mathrm{Th}} = \frac{e^{-\beta\omega\hat{a}^\dagger\hat{a}}}{Z} \qquad \beta = \frac{\hbar}{k_B T} \qquad Z = \frac{1}{1-e^{-\beta\omega}}$$

$$\hat{\rho}_{\mathrm{Th}} = \frac{1}{\pi Z}\int d^2\alpha\, e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}|\alpha\rangle\langle\alpha|\, e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}$$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$e^{-\frac{\beta\omega\hat{a}^\dagger\hat{a}}{2}}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-\frac{\beta\omega}{2}}\right)^n}{\sqrt{n!}}|n\rangle \qquad \left|e^{-\frac{\beta\omega}{2}}\alpha\right\rangle = e^{-\frac{1}{2}|\alpha|^2 e^{-\beta\omega}}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-\frac{\beta\omega}{2}}\right)^n}{\sqrt{n!}}|n\rangle$$

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_B T}-1}$$

$$\hat{\rho}_{\mathrm{Th}} = \frac{1}{\pi Z}\int d^2\alpha\, e^{-|\alpha|^2(1-e^{-\beta\omega})}\left|\alpha e^{-\frac{\beta\omega}{2}}\right\rangle\left\langle\alpha e^{-\frac{\beta\omega}{2}}\right|$$

$$= \frac{(1-e^{-\beta\omega})e^{\beta\omega}}{\pi}\int d^2\alpha' e^{-|\alpha'|^2 \alpha e^{\beta\omega}(1-e^{-\beta\omega})}|\alpha'\rangle\langle\alpha'| = \frac{1}{\pi\bar{n}}\int d^2\alpha\, e^{-\frac{|\alpha|^2}{\bar{n}}}|\alpha\rangle\langle\alpha| \equiv \int P_{\mathrm{Th}}(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha$$

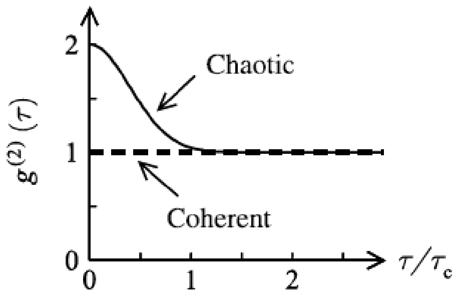
$$\alpha' = \alpha e^{-\frac{\beta\omega}{2}} \qquad d^2\alpha = e^{\beta\omega}d^2\alpha'$$

$$P_{\mathrm{Th}}(\alpha) = \frac{1}{\pi\bar{n}}\exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)$$

Classical light : $P_{\mathrm{Th}}(\alpha) > 0$ for all α

$P(\alpha)$ function for thermal light :

$$P_{\mathrm{Th}}(\alpha) = \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)$$



$$g^{(2)}(0) = \frac{\left\langle :\left(\hat{a}^\dagger \hat{a}\right)^2: \right\rangle}{\langle : \hat{a}^\dagger \hat{a} : \rangle} = \frac{\int P_{\mathrm{Th}}(\alpha)|\alpha|^4 d^2\alpha}{\left(\int P_{\mathrm{Th}}(\alpha)|\alpha|^2 d^2\alpha\right)^2} = 2$$

Photon bunching is thus predicted by quantum theory !

$$\int P_{\mathrm{Th}}(\alpha)|\alpha|^2 d^2\alpha = \frac{1}{\pi\bar{n}} \int \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right) |\alpha|^2 d^2\alpha$$

$\alpha = |\alpha| e^{i\theta}$ $\quad d^2\alpha = |\alpha| d\alpha d\theta$

$$= \frac{1}{\pi\bar{n}} 2\pi \int_0^\infty \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right) |\alpha|^3 d|\alpha|$$

$$= \frac{1}{\bar{n}} \int_0^\infty \exp\left(-\frac{u}{\bar{n}}\right) u du$$

$u = |\alpha|^2$ $\quad du = 2|\alpha| d|\alpha|$

$$= \bar{n} \int_0^\infty t e^{-t} dt = \bar{n}$$

$t = \frac{u}{\bar{n}}$ $\quad dt = \frac{du}{\bar{n}}$

$$\int P_{\mathrm{Th}}(\alpha)|\alpha|^4 d^2\alpha = \bar{n}^2 \int_0^\infty t^2 e^{-t} dt = 2\bar{n}^2$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$P_{\mathrm{Th}}(\alpha)$ function for thermal light also gives the intensity (&photon) fluctuations :

$$(\Delta n)^2 = \langle \hat{n} \rangle + (\Delta I)^2 = \bar{n} + \langle \hat{I}^2 \rangle - \langle \hat{I} \rangle^2 = \bar{n} + \bar{n}^2$$

Since as we saw on the previous page: $\langle \hat{I} \rangle = \int P_{\mathrm{Th}}(\alpha)|\alpha|^2 d^2\alpha = \bar{n}$ $\quad \langle \hat{I}^2 \rangle = \int P(\alpha)|\alpha|^4 d^2\alpha = 2\bar{n}^2$

This result, $(\Delta n)^2 = \bar{n} + \bar{n}^2$, is of course identical to that found directly with from density matrix of black-body radiation

The fact that, $\langle \hat{I}^2 \rangle - \langle \hat{I} \rangle^2 = \bar{n}^2 > 0$, tells us that black-body radiation doesn't require a quantum description.

$P(\alpha)$ of "quantum" light is more singular than a delta function (and/or somewhere negative):

The density matrix of a Fock state, $\hat{\rho}_n = |n\rangle\langle n|$

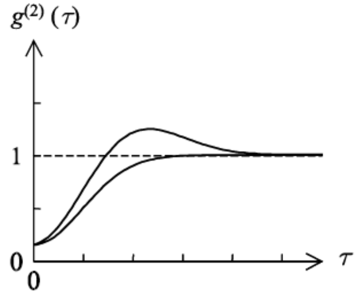
$$P_n(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}_n|u\rangle\, e^{\alpha u^* - u\alpha^*} d^2u$$

$$\langle -u|\hat{\rho}_n|u\rangle = \langle -u|\hat{\rho}_n|u\rangle = \langle -u|n\rangle\langle n|u\rangle = e^{-|u|^2}\frac{(-u^*u)^n}{n!}$$

$$|u\rangle = e^{-\frac{1}{2}|u|^2}\sum_{n=0}^{\infty}\frac{u^n}{\sqrt{n!}}|n\rangle$$

$$P_n(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2}\int\frac{(-u^*u)^n}{n!}e^{\alpha u^* - u\alpha^*}d^2u = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{1}{\pi^2}\int e^{\alpha u^* - u\alpha^*}d^2u = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{1}{\pi^2}\delta^{(2)}(\alpha)$$

When determining the $g^{(2)}(\tau)$ of quantum states it is often easier to return to a density matrix calculation:



$$g_n^{(2)}(0) = \frac{\int P_n(\alpha)|\alpha|^4 d^2\alpha}{\left(\int P_n(\alpha)|\alpha|^2 d^2\alpha\right)^2} = \frac{\left\langle :\left(\hat{a}^\dagger\hat{a}\right)^2: \right\rangle_{\hat{\rho}_n}}{\langle :\hat{a}^\dagger\hat{a}: \rangle_{\hat{\rho}_n}^2} \equiv \frac{\langle n|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|n\rangle}{\langle n|\hat{a}^\dagger\hat{a}|n\rangle^2} = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n}$$

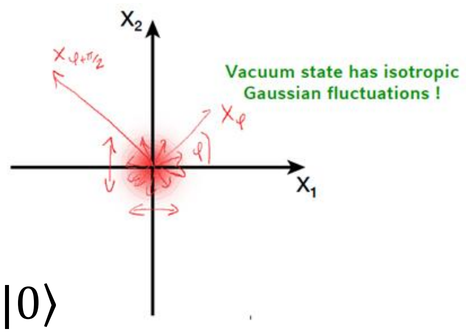
Squeezed states of light

The squeezing operator

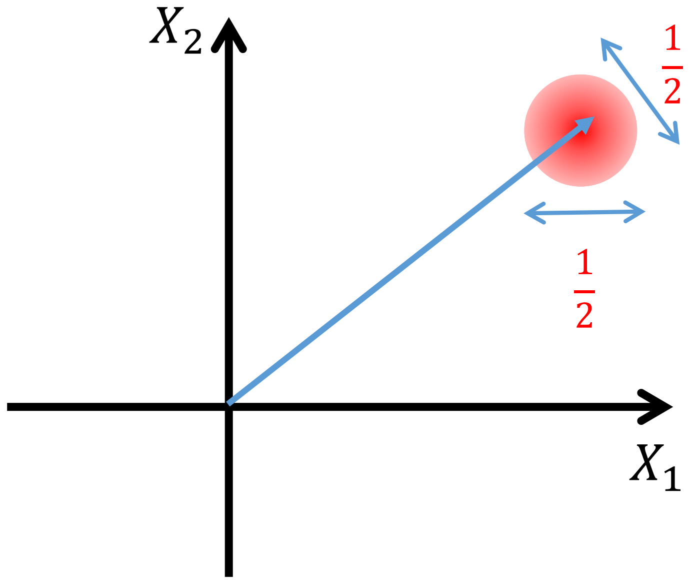
Displacement operator : $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$

$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$\alpha = |\alpha|e^{i\phi}$



$|0\rangle$



Squeezing operator : $\hat{S}(\xi) = \exp\left(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger,2}\right)$

$\xi = re^{i\theta}$

$|\psi_s\rangle = \hat{S}(\xi)|\psi\rangle$

Both $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ are unitary operators

$\hat{S}^\dagger(\xi) = \hat{S}(-\xi) = \hat{S}^{-1}(\xi)$

$\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = \hat{D}^{-1}(\alpha)$

The squeezing operator: $\hat{S}(\xi) = \exp(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger,2})$ $\quad \xi = re^{i\theta} \quad r \in \mathbb{R}$

Squeeze operator is unitary : $\hat{S}^\dagger(\xi) = \hat{S}(-\xi) = \hat{S}^{-1}(\xi)$

Any quantum state may be “squeezed” : $|\psi_s\rangle \equiv \hat{S}(\xi)|\psi\rangle$

$$\hat{A}_\xi \equiv \hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi) = \hat{a}\cosh r + e^{i\theta}\hat{a}^\dagger \sinh r$$

$$\hat{A}^\dagger{}_\xi \equiv \hat{S}(\xi)\hat{a}^\dagger\hat{S}^\dagger(\xi) = \hat{a}^\dagger\cosh r + e^{-i\theta}\hat{a} \sinh r$$

We can derive these relations by applying the Baker-Hausdorff Lemma :

$$e^{G\lambda}Ae^{-G\lambda} = A + \lambda[G,A] + \frac{\lambda^2}{2!}[G,[G,A]] + \ldots + \frac{\lambda^n}{n!}\underbrace{[G,[G,[G,\ldots[G,A]]]}_{(n \text{ times})} + \ldots]$$

The creation destruction operators satisfy the usual commutation relations : $[\hat{A}_\xi, \hat{A}^\dagger{}_\xi] = \cosh^2 r - \sinh^2 r = 1$

We can define a squeezed vacuum : $|0_s\rangle = \hat{S}(\xi)|0\rangle$

$$\hat{a}|0\rangle = 0 \Longrightarrow \hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi)\hat{S}(\xi)|0\rangle \Longrightarrow \hat{A}_\xi|0_s\rangle = 0$$

Uncertainties of the squeezed vacuum

$\xi = re^{i\theta}$

$$\hat{a} = \hat{A}_{\xi} \cosh r - e^{i\theta} \hat{A}^{\dagger}{}_{\xi} \sinh r$$

$$\hat{a}^{\dagger} = \hat{A}^{\dagger}{}_{\xi} \cosh r - e^{-i\theta} \hat{A}_{\xi} \sinh r$$

$$\hat{A}_{\xi} |0_s\rangle = 0$$

$$\hat{X}_1 \equiv \frac{1}{2}\left(\hat{a} + \hat{a}^{\dagger}\right)$$

$$\hat{X}_2 \equiv \frac{1}{2i}\left(\hat{a} - \hat{a}^{\dagger}\right)$$

$$\left\langle \hat{X}_1 \right\rangle_{0_s} = \frac{1}{4} \langle 0_s | \left(\hat{A}^{\dagger}{}_{\xi} \cosh r - e^{i\theta} \hat{A}^{\dagger}{}_{\xi} \sinh r + \hat{A}^{\dagger}{}_{\xi} \cosh r - e^{-i\theta} \hat{A}_{\xi} \sinh r \right) |0_s\rangle = 0$$

$$\left\langle \hat{X}_2 \right\rangle_{0_s} = \frac{1}{4} \langle 0_s | \left(\hat{A}^{\dagger}{}_{\xi} \cosh r - e^{i\theta} \hat{A}^{\dagger}{}_{\xi} \sinh r - \hat{A}^{\dagger}{}_{\xi} \cosh r + e^{-i\theta} \hat{A}_{\xi} \sinh r \right) |0_s\rangle = 0$$

Uncertainties of the squeezed vacuum

$$\xi = re^{i\theta}$$

$$\hat{a} = \hat{A}_\xi \cosh r - e^{i\theta}\hat{A}^\dagger{}_\xi \sinh r$$

$$\hat{a}^\dagger = \hat{A}^\dagger{}_\xi \cosh r - e^{-i\theta}\hat{A}_\xi \sinh r$$

$$\hat{A}_\xi|0_s\rangle = 0$$

$$\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 \equiv \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{X}_1^2 \equiv \frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a} + 1)$$

$$\hat{X}_2^2 \equiv -\frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}) = -\frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a} - 1)$$

$$[\hat{a}, \hat{a}^\dagger] = 1 \Rightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$$

$$\langle\hat{X}_1^2\rangle_{0_s} \equiv \frac{1}{4}\langle 0_s|\left(2\hat{A}^\dagger{}_\xi\hat{A}_\xi\cosh^2 r + 2\hat{A}_\xi\hat{A}^\dagger{}_\xi\sinh^2 r - e^{i\theta}\left[\hat{A}_\xi\hat{A}^\dagger{}_\xi + \hat{A}^\dagger{}_\xi\hat{A}_\xi\right]\sinh r\cosh r - e^{-i\theta}\left[\hat{A}_\xi\hat{A}^\dagger{}_\xi + \hat{A}^\dagger{}_\xi\hat{A}_\xi\right]\sinh r\cosh r + 1 + \cdots\right)|0_s\rangle$$

$$= \frac{1}{4}(\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta)$$

$$\langle\hat{X}_1^2\rangle_{0_s} = \frac{1}{4}(\cosh^2 r + \sinh r + 2\sinh r\cosh r\cos\theta)$$

Simpler expressions when taking $\theta = 0 : \xi = re^{i\theta} \to r$

$$\hat{a} = \hat{A}_\xi \cosh r - \hat{A}^\dagger{}_\xi \sinh r$$

$$\hat{a}^\dagger = \hat{A}^\dagger{}_\xi \cosh r - \hat{A}_\xi \sinh r$$

$$\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 \equiv \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{A}_\xi|0_s\rangle = 0$$

$$\langle \hat{X}_1^2 \rangle_{0_s} = \frac{1}{4}(\cosh^2 r + \sinh^2 r - 2\sinh r \cosh r) = \frac{1}{4}(\cosh r - \sinh r)^2 = \frac{1}{4}e^{-2r}$$

$$\langle \hat{X}_2^2 \rangle_{0_s} = \frac{1}{4}(\cosh^2 r + \sinh r + 2\sinh r \cosh r) = \frac{1}{4}(\cosh r + \sinh r)^2 = \frac{1}{4}e^{2r}$$

$$\cosh r = \frac{e^r + e^{-r}}{2}$$

$$\sinh r = \frac{e^r - e^{-r}}{2}$$

Since $\langle \hat{X}_1 \rangle_{0_s} = \langle \hat{X}_2 \rangle_{0_s} = 0$

$$(\Delta X_1)^2_{0_s} = \langle \hat{X}_1^2 \rangle_{0_s} = \frac{1}{4}e^{-2r}$$

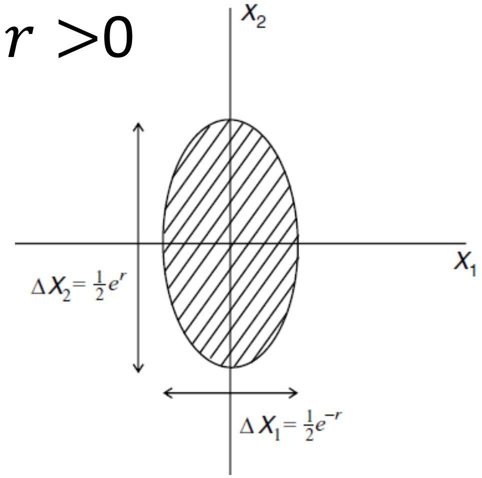
$$(\Delta X_2)^2_{0_s} = \langle \hat{X}_2^2 \rangle_{0_s} = \frac{1}{4}e^{2r}$$

Quadrature squeezing with $\xi = r$

$$\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \quad \hat{X}_2 \equiv \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{A}_\xi |0_s\rangle = 0$$

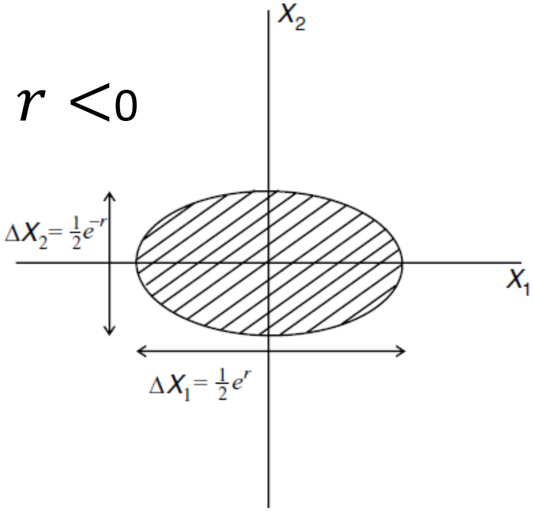
$r > 0$



$$\Delta X_1 < \frac{1}{2} \qquad \Delta X_2 > \frac{1}{2}$$

$$(\Delta X_1)^2_{0_s} = \frac{1}{4}e^{-2r} \qquad (\Delta X_2)^2_{0_s} = \frac{1}{4}e^{2r}$$

$r < 0$



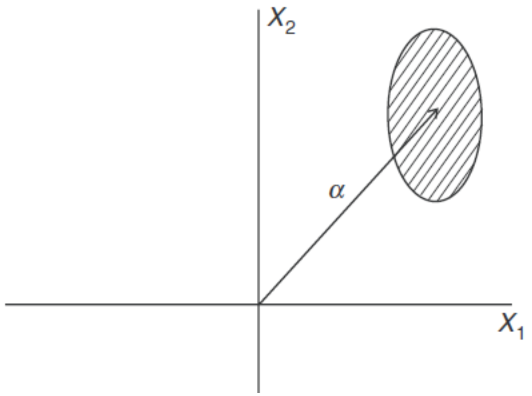
$$\Delta X_1 > \frac{1}{2} \qquad \Delta X_2 < \frac{1}{2}$$

We see that the coherent vacuum satisfies the minimal uncertainty relation so no quantum rules are violated :

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

Displaced squeezed vacuum state: $|\alpha, \xi\rangle \equiv \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$

$$\hat{X}_1 \equiv \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{X}_2 \equiv \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)$$

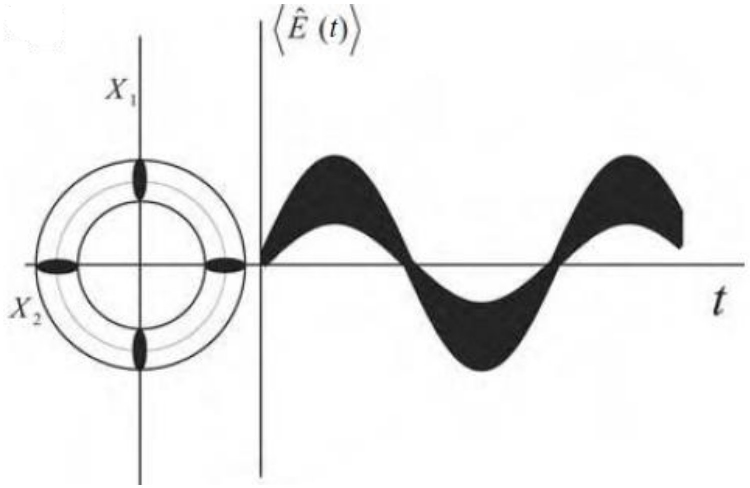


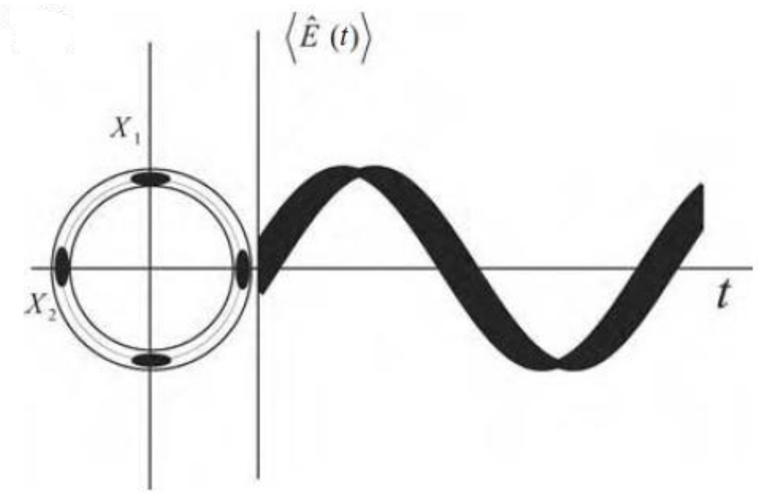
$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

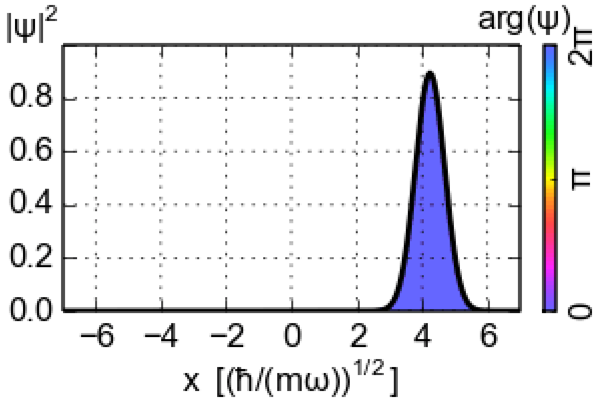
The displaced squeezed state can have many photons with minimal uncertainty and $\Delta X < \frac{1}{2}$

Often called a coherent or “ideal” squeezed state

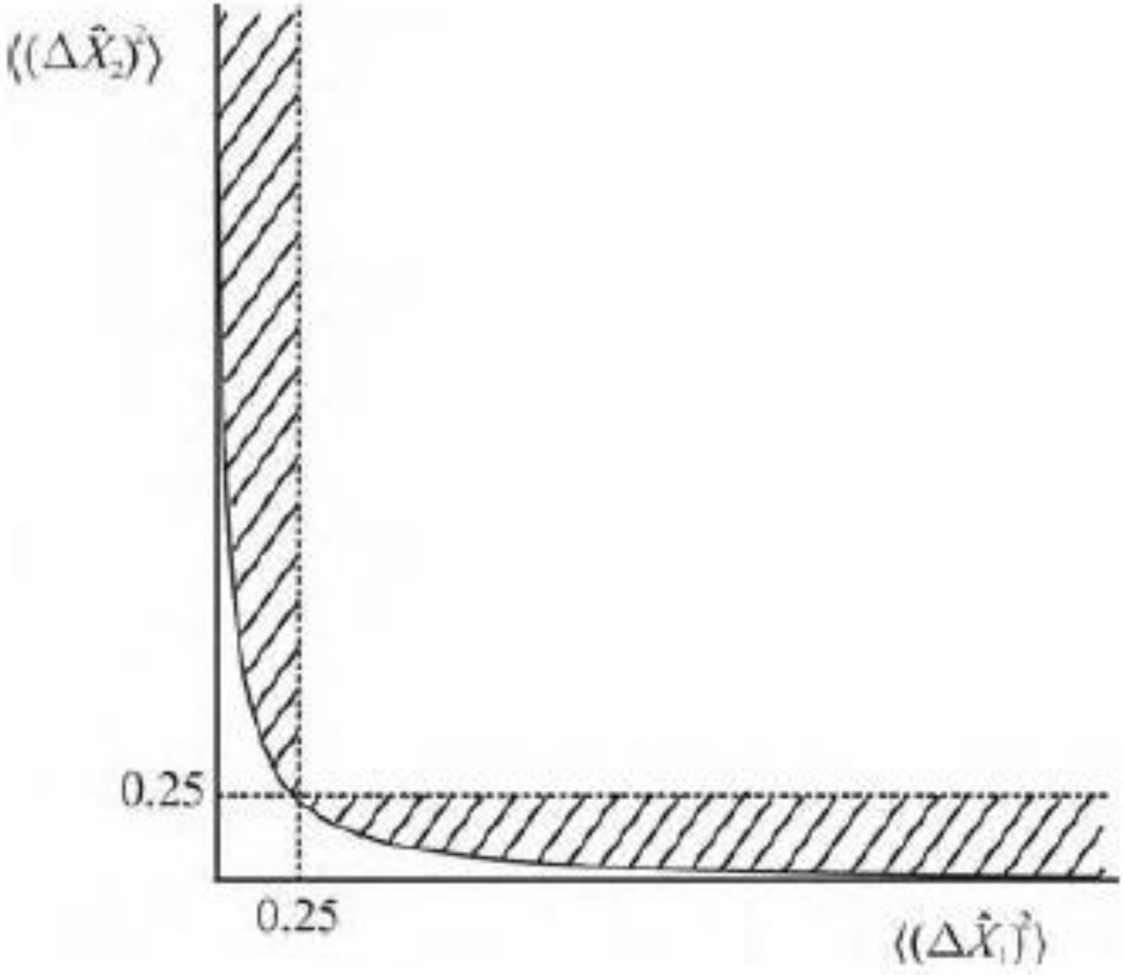
Time evolution of coherent squeezed states:







Domains of squeezing



Squeezed states are necessarily non-classical !

$\hat{X}_1 \equiv \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ $\quad$ $\hat{X}_2 \equiv \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$ $\qquad$ $|\alpha, \xi\rangle \equiv \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$

$$(\Delta X_1)^2 = \left\langle \left(\hat{X}_1 - \langle \hat{X}_1 \rangle\right)^2 \right\rangle = \frac{1}{4} + \left\langle : \left(\hat{X}_1 - \langle \hat{X}_1 \rangle\right)^2 : \right\rangle = \frac{1}{4}\left\{1 + \int P(\alpha) d\alpha^2 \left[\alpha + \alpha^* - \left(\langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle\right)\right]^2\right\}$$

$$(\Delta X_2)^2 = \left\langle \left(\hat{X}_2 - \langle \hat{X}_2 \rangle\right)^2 \right\rangle = \frac{1}{4} + \left\langle : \left(\hat{X}_1 - \langle \hat{X}_1 \rangle\right)^2 : \right\rangle = \frac{1}{4}\left\{1 + \int P(\alpha) d\alpha^2 \left[\frac{\alpha - \alpha^*}{i} - \left(\frac{\langle \hat{a} \rangle - \langle \hat{a}^\dagger \rangle}{i}\right)\right]^2\right\}$$

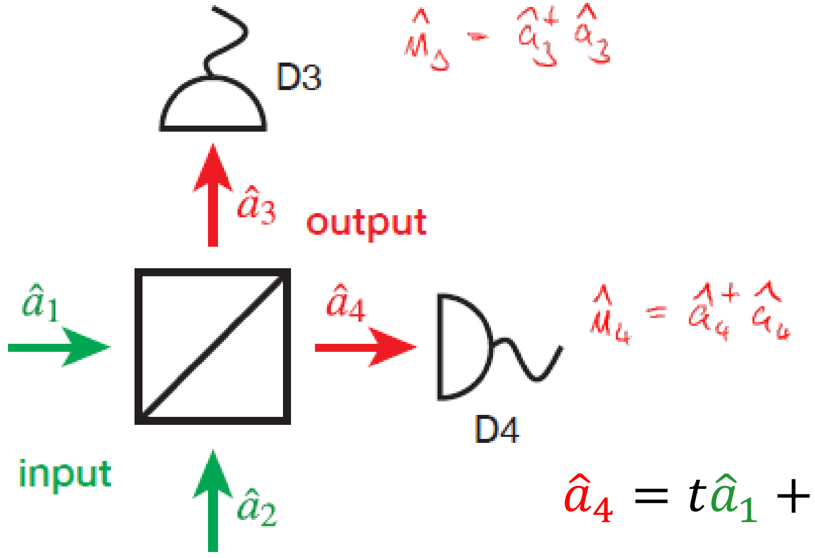
$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

Since the terms in red square brackets [ ] are positive definite then $(\Delta X_i)^2 < \frac{1}{4}$ requires $P(\alpha) < 0$ or $P(\alpha)$ more singular than a delta function.

This is the accepted requirement for light to require a quantum description !

Balanced homodyne detection

Detectors in BS Outputs (two input channels)



$$\hat{a}_4 = t\hat{a}_1 + r\hat{a}_2$$

$$\hat{N}_4 = \hat{a}_4^\dagger \hat{a}_4 = \left(T^* \hat{a}_1^\dagger + R^* \hat{a}_2^\dagger\right)\left(T\hat{a}_1 + R\hat{a}_2\right)$$

$$= \frac{1}{2}\left(\hat{a}_1^\dagger - i\hat{a}_2^\dagger\right)\left(\hat{a}_1 + i\hat{a}_2\right)$$

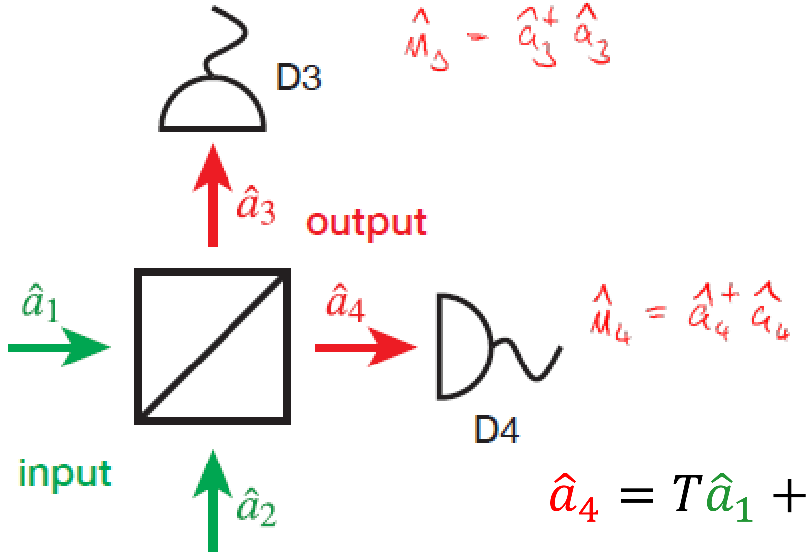
$$= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 + i\hat{a}_1^\dagger \hat{a}_2 - i\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$$

Balanced 50/50 Beam-splitter

$$T = \frac{1}{\sqrt{2}} \qquad R = \frac{i}{\sqrt{2}}$$

$$T = |T| \qquad R = i|R|$$

Output photon number difference



$$\hat{N}_4 = \hat{a}_4^\dagger \hat{a}_4 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 + i\hat{a}_1^\dagger \hat{a}_2 - i\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$$

$$\hat{N}_3 = \hat{a}_3^\dagger \hat{a}_3 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - i\hat{a}_1^\dagger \hat{a}_2 + i\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2$$

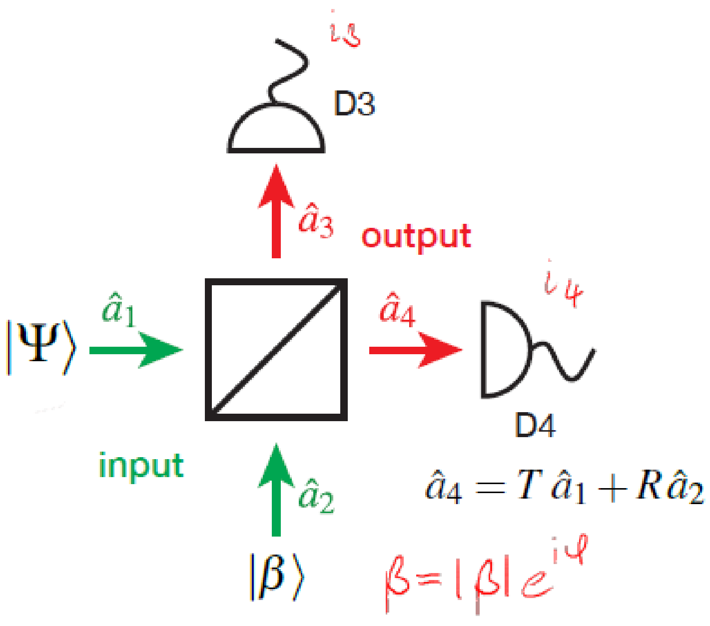
$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{N}_3 - \hat{N}_4 = i\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)$$

Balanced 50/50 Beam-splitter

$$T = \frac{1}{\sqrt{2}} \qquad R = \frac{i}{\sqrt{2}}$$

Detection of $|\Psi\rangle_1$ with coherent state $|\beta\rangle_2$ as a "homodyne" reference beam



$$\hat{N}_4 = \hat{a}_4^\dagger \hat{a}_4 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 + i\hat{a}_1^\dagger \hat{a}_2 - i\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$$

$$\hat{N}_3 = \hat{a}_3^\dagger \hat{a}_3 = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - i\hat{a}_1^\dagger \hat{a}_2 + i\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right)$$

$$i_{34} = i_3 - i_4 \propto \langle \text{in}|\hat{N}_3 - \hat{N}_4|\text{in}\rangle$$

$$= -2\ {}_2\langle\beta|\ {}_1\langle\Psi|\frac{1}{i2}\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)|\Psi\rangle_1|\beta\rangle_2$$

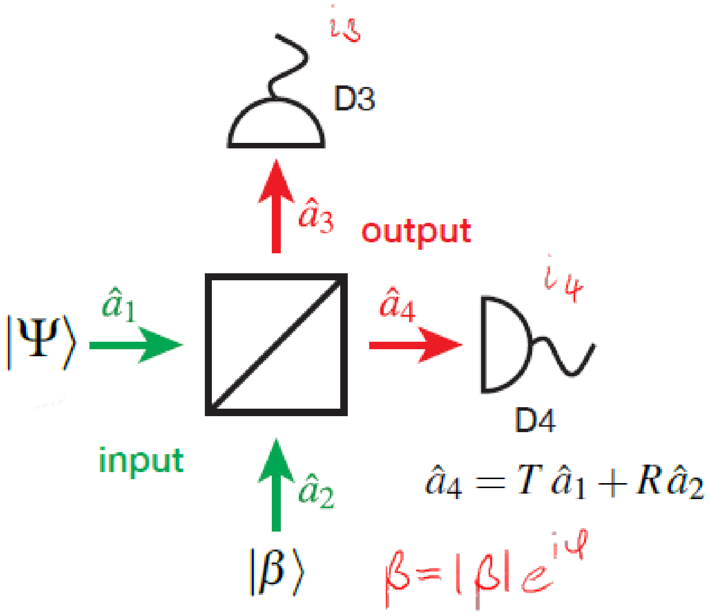
$$= -2\ {}_2\langle\beta|\ {}_1\langle\Psi|\frac{1}{2i}\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)|\Psi\rangle_1|\beta\rangle_2$$

Input State

$$|\text{in}\rangle = |\Psi\rangle_1|\beta\rangle_2$$

$$\beta = |\beta|e^{i\varphi}$$

Output photon number difference



$$i_{34} = i_3 - i_4 \propto \langle \text{in}|\hat{N}_3 - \hat{N}_4|\text{in}\rangle = -2\ {}_2\langle\beta|\ {}_1\langle\Psi|\frac{1}{2i}\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)|\Psi\rangle_1|\beta\rangle_2$$

$$= -2|\beta|\ {}_2\langle\beta|\ {}_1\langle\Psi|\frac{1}{2i}\left(\hat{a}_1 e^{-i\varphi} - \hat{a}_1^\dagger e^{i\varphi}\right)|\Psi\rangle_1|\beta\rangle_2$$

$$= -2|\beta|\ {}_2\langle\beta||\beta\rangle_2\ {}_1\langle\Psi|\left(\hat{X}_{\varphi+\frac{\pi}{2}}\right)|\Psi\rangle_1$$

$$= -2|\beta|\ {}_1\langle\Psi|\left(\hat{X}_{\varphi+\frac{\pi}{2}}\right)|\Psi\rangle_1 = -2|\beta|\left\langle\hat{X}_{\varphi+\frac{\pi}{2}}\right\rangle_\Psi$$

Input State $|\text{in}\rangle = |\Psi\rangle_1|\beta\rangle_2$

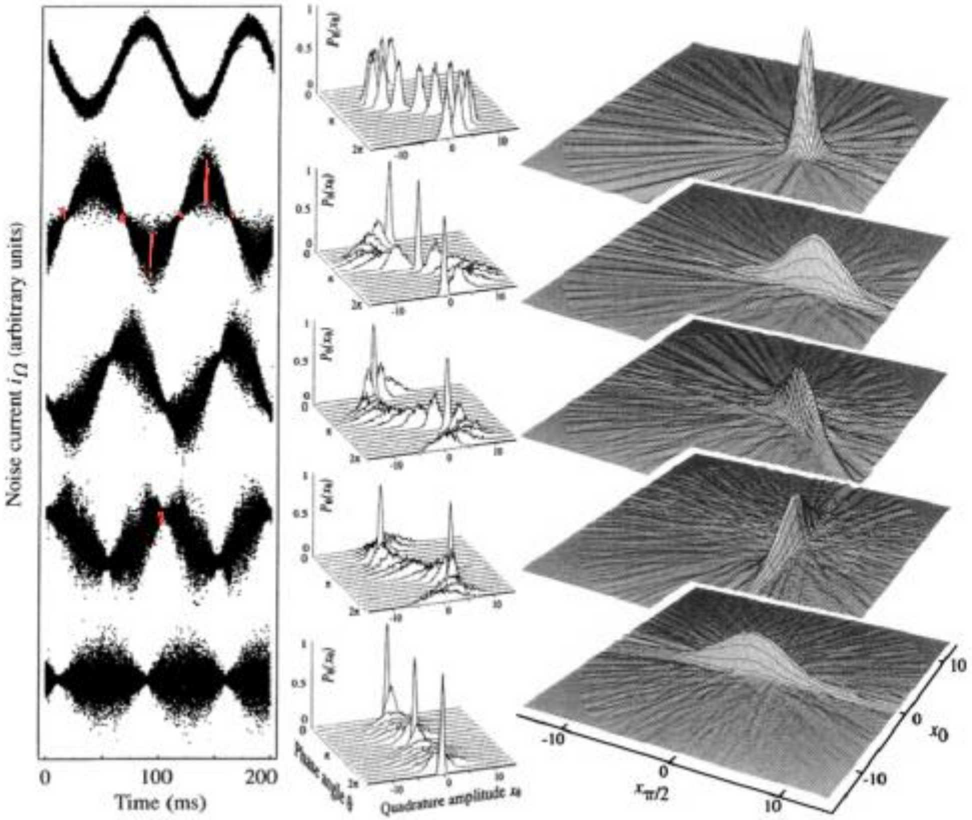
$\beta = |\beta|e^{i\varphi}$

$$\hat{X}_{\varphi+\frac{\pi}{2}} = \frac{1}{2i}\left(\hat{a}e^{-i\varphi} - \hat{a}^\dagger e^{i\varphi}\right) = \frac{1}{2i}\left(\hat{a}\cos\varphi - i\,\hat{a}\sin\varphi - \hat{a}^\dagger\cos\varphi - i\hat{a}^\dagger\sin\varphi\right)$$

$$= \left[\frac{(\hat{a} - \hat{a}^\dagger)}{2i}\cos\varphi - \frac{(\hat{a} + \hat{a}^\dagger)}{2}\sin\varphi\right] = \left[\hat{X}_2\cos\varphi - \hat{X}_1\sin\varphi\right]$$

Measuring the Quadrature Operator

$\hat{X}_{1,\varphi+\frac{\pi}{2}}$



Coherent State

Phase Squeezed State

Amplitude Squeezed State

Vacuum Squeezed State

G. Breitenbach et al. Nature 387, 471-475 (1997)

Statistics of the squeezed vacuum

$$\hat{a} = \hat{A}_\xi \cosh r - e^{i\theta}\hat{A}^\dagger{}_\xi \sinh r$$

$$\hat{a}^\dagger = \hat{A}^\dagger{}_\xi \cosh r - e^{-i\theta}\hat{A}_\xi \sinh r$$

$$\xi = re^{i\theta}$$

$$\hat{A}_\xi|0_s\rangle = 0$$

$$\langle\hat{N}\rangle_{0_s} = \langle 0_s|\hat{a}^\dagger\hat{a}|0_s\rangle = \langle 0_s|\left(\hat{A}^\dagger{}_\xi \cosh r - e^{-i\theta}\hat{A}_\xi \sinh r\right)\left(\hat{A}_\xi \cosh r - e^{i\theta}\hat{A}^\dagger{}_\xi \sinh r\right)|0_s\rangle$$

$$= \langle 0_s| -e^{-i\theta}\hat{A}_\xi \sinh r\left(-e^{i\theta}\hat{A}^\dagger{}_\xi \sinh r\right)|0_s\rangle = \sinh^2 r\langle 0_s|\hat{A}_\xi\hat{A}^\dagger{}_\xi|0_s\rangle = \sinh^2 r$$

$$\langle\hat{N}\rangle_{0_s} = \sinh^2 r$$

Since $\langle 0_s|\hat{A}_\xi\hat{A}^\dagger{}_\xi|0_s\rangle = \langle 1_s|1_s\rangle = 1$ alternatively use the commutator to :put things in normal order $\langle 0_s|\hat{A}_\xi\hat{A}^\dagger{}_\xi|0_s\rangle = \langle 0_s|1 + \hat{A}^\dagger{}_\xi\hat{A}_\xi|0_s\rangle = 1$

Statistics of the squeezed vacuum

$$\hat{a} = \hat{A}_\xi \cosh r - e^{i\theta}\hat{A}^\dagger{}_\xi \sinh r \equiv \hat{A}_\xi\ \mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger{}_\xi \mathrm{sh}_r$$

$$\hat{a}^\dagger = \hat{A}^\dagger{}_\xi \cosh r - e^{-i\theta}\hat{A}_\xi \sinh r \equiv \hat{A}^\dagger{}_\xi \mathrm{ch}_r - e^{-i\theta}\hat{A}_\xi \mathrm{sh}_r$$

$\xi = re^{i\theta}$

$\hat{A}_\xi|0_s\rangle = 0$

$\bar{n}_{0_s} = \sinh^2 r$

$$\langle\hat{N}^2\rangle_{0_s} = \langle 0_s|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|0_s\rangle = \langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle + \langle 0_s|\hat{a}^\dagger\hat{a}|0_s\rangle = \langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle + \langle\hat{N}\rangle_{0_s} = \langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle + \sinh^2 r$$

$$\langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle = \langle 0_s|\left(\mathrm{ch}_r\hat{A}^\dagger - e^{-i\theta}\mathrm{sh}_r\hat{A}\right)\left(\hat{A}^\dagger\mathrm{ch}_r - e^{-i\theta}\hat{A}\mathrm{sh}_r\right)\left(\hat{A}\mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger\mathrm{sh}_r\right)\left(\hat{A}\mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger\mathrm{sh}_r\right)|0_s\rangle$$

$$= \mathrm{sh}_r^2\langle 0_s|\hat{A}\left(\mathrm{ch}_r\hat{A}^\dagger - e^{-i\theta}\hat{A}\mathrm{sh}_r\right)\left(\hat{A}\mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger\mathrm{sh}_r\right)\hat{A}^\dagger|0_s\rangle = \mathrm{sh}_r^2\langle 1_s|\left(\mathrm{ch}_r\hat{A}^\dagger - e^{-i\theta}\hat{A}\mathrm{sh}_r\right)\left(\hat{A}\mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger\mathrm{sh}_r\right)|1_s\rangle$$

$$\left(\mathrm{ch}_r\hat{A}^\dagger - e^{-i\theta}\hat{A}\mathrm{sh}_r\right)\left(\hat{A}\mathrm{ch}_r - e^{i\theta}\hat{A}^\dagger\mathrm{sh}_r\right) = \left(\mathrm{ch}_r^2\hat{A}^\dagger\hat{A} - e^{i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}^\dagger\hat{A}^\dagger - e^{-i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}\hat{A} + \mathrm{sh}_r^2\hat{A}\hat{A}^\dagger\right)$$

$$= \left[\mathrm{ch}_r^2\hat{A}^\dagger\hat{A} - e^{i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}^\dagger\hat{A}^\dagger - e^{-i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}\hat{A} + \mathrm{sh}_r^2\left(1 + \hat{A}^\dagger\hat{A}\right)\right]$$

$$\langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle = \mathrm{sh}_r^2\langle 1_s|\mathrm{ch}_r^2\hat{A}^\dagger\hat{A} - e^{i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}^\dagger\hat{A}^\dagger - e^{-i\theta}\mathrm{sh}_r\mathrm{ch}_r\hat{A}\hat{A} + \mathrm{sh}_r^2\left(1 + \hat{A}^\dagger\hat{A}\right)|1_s\rangle = \mathrm{sh}_r^2(\mathrm{ch}_r^2 + 2\mathrm{sh}_r^2) = \mathrm{sh}_r^2(1 + 3\mathrm{sh}_r^2)$$

$$\langle\hat{N}^2\rangle_{0_s} = \langle 0_s|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|0_s\rangle + \sinh^2 r = 3\sinh^4 r + 2\sinh^2 r$$

$$\mathrm{Var}(N) \equiv \langle\hat{N}^2\rangle_{0_s} - \bar{n}_{0_s}^2 = 2\bar{n}_{0_s}\left(\bar{n}_{0_s} + 1\right) > \bar{n}_{0_s}$$

Statistics of the squeezed vacuum

$\xi = re^{i\theta}$

$$\Delta N_{0_s}^2 = \langle \hat{N}^2 \rangle_{0_s} - \langle \hat{N} \rangle_{0_s}^2 = 2\bar{n}_{0_s}(\bar{n}_{0_s} + 1) > \langle \hat{N} \rangle_{0_s}$$

$\hat{A}_\xi |0_s\rangle = 0$

squeezed vacuum

Poissonian distribution

$$\langle \hat{N} \rangle_{0_s} = \sinh^2 r = \bar{n}_{0_s} \qquad \langle \Delta\hat{N} \rangle_{0_s} = \sqrt{2\bar{n}_{0_s}(\bar{n}_{0_s} + 1)}$$

Ordinary vacuum :

$$\langle \hat{N} \rangle_0 = 0 \qquad \langle \Delta\hat{N} \rangle_0 = 0$$

$$\langle \Delta\hat{N} \rangle_{0_s} \geq \sqrt{\bar{n}_{0_s}}$$

Coherent state :

Super-Poissonian distribution for $r > 0$!

$$\langle \hat{N} \rangle_\alpha = |\alpha|^2 = \bar{n} \qquad \langle \Delta\hat{N} \rangle_\alpha = |\alpha| = \sqrt{\bar{n}_\alpha}$$

$$\langle \Delta\hat{N} \rangle_\alpha = \sqrt{\bar{n}_\alpha}$$

Equality is only achieved when $r = 0$!